Test 1, TTVN Math Methods

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Here is the first test for Math Methods. Please show all your work, and work independently, but you may use any reference materials.

This test is due Tuesday, Feb. 27.

1. Suppose we want to find a dimensionless number related to animal locomotion. An animal walks across the ground with legs of a certain length, and swings the legs forward and backward at least somewhat like a pendulum, so the acceleration of gravity will be important. The animal walking with a given type of gait will have some optimum speed for that gait.

   Use the physical parameters of L, ν, and g to find a dimensionless number for this situation. The number you will get is known as the *Froude number* Fr.

   The Froude number is also important as a scaling parameter in other contexts involving a length, speed, and the acceleration of gravity, such as naval architecture and geophysical wave phenomena.

2. Find the components of both the velocity and acceleration vectors in cylindrical coordinates.

3. For complex numbers, **Euler's formula** is

   $$e^{i\theta} = \cos\theta + i\sin\theta$$

   where θ is a real number. This relation is used in the definition of the exponential function for complex numbers. Some motivation is given by the Taylor series.

   (a) Show that the Taylor series of the left hand side of the above expression is equal to the Taylor series of the right hand side. Recall that $i^2 = -1$.

   (b) Show that $e^{-i\theta} = \cos\theta - i\sin\theta$.

   (c) Show that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

4. The total energy of a particle is found in special relativity to be

   $$E = \gamma mc^2$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

c is the speed of light, and v is the speed of the particle as measured from some reference frame.

Expand E in a Taylor series and show that for $v \ll c$ that

$$E = mc^2 + \frac{1}{2}mv^2$$

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5. (a) Consider the vector function

$$\vec{\mathbf{v}} = \frac{1}{r^2}\hat{\mathbf{r}}$$

Calculate its divergence using the definition of $\nabla \cdot \vec{\mathbf{v}}$. The answer may surprise you.

(b) The divergence theorem (aka Gauss law) says

$$\oint_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{S}} = \oint_V (\nabla \cdot \vec{\mathbf{v}}) dV.$$

The left side is a flux integral. Calculate the value of the left side for the function above using a sphere of radius R centered on the origin for your surface.

(c) Explain what is going on here. You may wish to consult chapters 8 and/or 14.

6. Consider the cylindrical flow

$$\vec{\mathbf{v}} = \hat{\phi} v(r),$$

with

$$v(r) = \frac{A}{r}.$$

Show that the vorticity vanishes for this flow.