Some residue integration examples

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Here are a couple of examples of contour integration using residues and a contour in the upper half-plane. They are exercises from Complex variables, harmonic and analytic functions, by Francis J. Flanigan.

1 First example

Integrate
\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}. \] (1)

Use the complex function
\[ f(z) = \frac{1}{z^2 + z + 1}. \] (2)

We will use a contour that goes from $-R$ to $R$ on the real axis, and then close it with a semicircular contour in the upper half of the complex plane, so $C = C_{\text{real}} + C_R$. We let $R \to \infty$ to get the whole real axis.

First we have to make sure that along the semicircular $C_R$, the contribution of the integral vanishes. We use Flanigan's "lemma" which is that the integral over $C_R$ vanishes if
\[ \lim_{R=|z| \to \infty} zf(z) = 0. \]

So consider
\[ \lim_{R=|z| \to \infty} \frac{z}{z^2 + z + 1} = \lim_{|z| \to \infty} \frac{1}{z + 1 + 1/z} = 0, \] (3)

which satisfies the lemma.\(^1\) So the integral along the real axis will be given by the residue of the pole in the upper half-plane.

The denominator of $f(z)$ has roots at
\[ z = -1 \pm \frac{i\sqrt{3}}{2} \]

\(^1\)This is based on the ML-inequality or "estimation lemma", see Wikipedia, etc.
So
\[ f(z) = \frac{1}{(z - \frac{1}{2}(-1 - i\sqrt{3}))(z - \frac{1}{2}(-1 + i\sqrt{3}))}, \quad (4) \]
call \( z_0 = \frac{1}{2}(-1 + i\sqrt{3}) \), this is the location of the pole in the upper half-plane, which will be enclosed by the contour. The pole is simple so we can find the residue using
\[ \text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z). \quad (5) \]
The residue is
\[ \text{Res}(f, z_0) = \lim_{z \to z_0} \frac{(z - z_0)}{(z - \frac{1}{2}(-1 - i\sqrt{3}))(z - z_0)} = \left[ \frac{1}{2}(-1 + i\sqrt{3} + 1 + i\sqrt{3}) \right]^{-1} = \frac{1}{i\sqrt{3}}. \quad (6), (7), (8) \]
So
\[ \int_C f(z) \, dz = \int_{-\infty}^{\infty} \frac{dx}{x^2 + x + 1}, \quad (9) \]
\[ = 2\pi i \text{Res}(f, z_0), \quad (10) \]
\[ = 2\pi i \left( \frac{1}{i\sqrt{3}} \right), \quad (11) \]
\[ = \frac{2\pi}{\sqrt{3}}. \quad (12) \]

2 Slightly more complicated example.

Integrate
\[ \int_{-\infty}^{\infty} \frac{x - 2^{1/3}}{x^3 - 2} \, dx, \quad (13) \]
where \( 2^{1/3} \) is the real cube root of 2. (Remember there are \( n \)th roots of a number, some of which are complex.)

We are going to use the same contour as before. First, check that the integral will vanish along \( C_R \):
\[ \lim_{|z| \to \infty} zf(z) = \lim_{|z| \to \infty} \frac{z(z - 2^{1/3})}{z^3 - 2} \]
\[ = \lim_{|z| \to \infty} \left[ \frac{z^2}{z^3 - 2} - \frac{2^{1/3}z}{z^3 - 2} \right] = 0, \quad (14), (15) \]
so we don't have to worry about $C_R$. Now what are the poles of

$$f(z) = \frac{z - 2^{1/3}}{z^3 - 2}.$$  \hfill (16)

They will be the zeros of $z^3 - 2$, i.e. the three cube roots of 2, which are

$$2^{1/3}, 2^{1/3}e^{i2\pi/3}, 2^{1/3}e^{i4\pi/3}.$$  

The numerator of $f(z)$ will eliminate the first, which is on the real axis. The second will be in the upper half-plane, so it is inside $C$, the last is in the lower half-plane.

Now follow the procedure:

$$f(z) = \frac{(z - 2^{1/3})}{(z - 2^{1/3})(z - 2^{1/3}e^{i2\pi/3})(z - 2^{1/3}e^{i4\pi/3})}.$$  \hfill (17)

$$= \frac{1}{(z - 2^{1/3}e^{i2\pi/3})(z - 2^{1/3}e^{i4\pi/3})}.$$  \hfill (18)

Our enclosed pole is at $z_0 = 2^{1/3}e^{i2\pi/3}$, so we can get

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{z - z_0}{(z - 2^{1/3}(z - 2^{1/3}e^{i4\pi/3})}, \hfill (19)
\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(z - 2^{1/3}e^{i4\pi/3})}, \hfill (20)
\text{Res}(f, z_0) = \left(2^{1/3}(e^{i2\pi/3} - e^{i4\pi/3})\right)^{-1}.$$  \hfill (21)

working out the real and imaginary parts of the exponentials, get

$$\text{Res}(f, z_0) = \frac{1}{2^{1/3}3^{1/2}i}.$$  \hfill (22)

Now we put this all together and

$$\int_{-\infty}^{\infty} \frac{x - 2^{1/3}}{x^3 - 2} \, dx = 2\pi i \text{Res}(f, z_0),$$  \hfill (23)
$$= \frac{2\pi}{2^{1/3}3^{1/2}}.$$  \hfill (24)

This looks like it would be very difficult to obtain using real variable methods.

\footnote{If this is confusing, review the properties of complex roots. In any case, it is a good idea to sketch a diagram of the root locations.}