Shortest distance: minimization of curvature

Math Methods, D. Craig

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Section 10.2

Near a minimum of a 1–D function, the function changes very little if we move a small ϵ away from the minimum—it is a stationary point.

Now we seek to find a stationary *function*, given certain conditions. In this case we are going to find the minimum distance (function) between two points. How can we "move" away from a function? Add a little variation function, and linearize with respect to it.

Problem a

$$L[h] = \int_a^b \sqrt{dx^2 + dy^2} \quad (1)$$

$$= \int_a^b \sqrt{dx^2 + \left(\frac{dh}{dx}\right)^2 dx^2} \quad (2)$$

$$= \int_a^b \sqrt{1 + \left(\frac{dh}{dx}\right)^2}\, dx \quad (3)$$

$$= \int_a^b \sqrt{1 + h_x^2}\, dx. \quad (4)$$

$L[h]$ is a quantity that depends on the form of a function—a "functional."

When $L[h]$ is a minimum, it does not change to first order when $h(x)$ is perturbed. Add a function $\epsilon(x)$ to $h(x)$ which vanishes at the end-points: $\epsilon(a) = \epsilon(b) = 0$.

Problem b:

Put in $h \to h+\epsilon$, and expand in a Taylor series. Write

$$\frac{d}{dx}(h+\epsilon) = \frac{dh}{dx} + \frac{d\epsilon}{dx} = h_x + \epsilon_x.$$

We want to get

$$\delta L[h] = L[h+\epsilon] - L[h] \tag{5}$$

The variation of the integrand (see eq. 3.18) is

$$\delta\left[\sqrt{1+h_x^2}\right] \approx \epsilon_x \frac{d}{dh_x}\left(\sqrt{1+h_x^2}\right) = \frac{h_x \epsilon_x}{\sqrt{1+h_x^2}}. \tag{6}$$

So the variation of the integral is

$$\delta L[h] = \int_a^b \frac{h_x \epsilon_x}{\sqrt{1+h_x^2}}\, dx \tag{7}$$

Now your text makes the simplifying assumption that $h_x \ll 1$, so*

$$\delta L[h] = \int_a^b h_x \epsilon_x \, dx \tag{8}$$

*This is just to simplify the algebra. The analysis can be continued with an uglier integral.

Problem c:

Integrate this by parts:

$$\int_a^b h_x \epsilon_x \, dx = h_x \epsilon_x \Big|_a^b - \int_a^b \frac{d^2 h}{dx^2} \epsilon \, dx \tag{9}$$

Using $\epsilon(a) = \epsilon(b) = 0$ this is

$$\delta L[h] = -\int_a^b \frac{d^2 h}{dx^2} \epsilon(x) \, dx. \tag{10}$$

This must vanish for *any* $\epsilon(x)$ if $L[h]$ is stationary. So

$$\frac{d^2 h}{dx^2} = 0, \tag{11}$$

which means the curvature vanishes, hence a straight line is the shortest distance between two points.

This procedure is not just beating the obvious to death. Notice that we used

$$dr = \sqrt{dx^2 + dy^2}$$

to define the our little increment of line length. This is a distance function or *metric.*

If we were trying to find the minimum distance on a sphere in 2–D, or some other curved space, we could use a different distance function and follow the same procedure to find stationary paths.

This idea is very important in the geometry of curved spaces, including General Relativity.