Laplace's equation, harmonic functions

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This introduction follows part of Ch 1. of Wyld, *Mathematical Methods for Physics*, 1976.

Heat conduction

Suppose we have heat flowing through a conductive medium. The flux of energy $\vec{\mathbf{F}}$ is related to the gradient of the temperature by the thermal conductivity:

$$\vec{\mathbf{F}} = -K(\nabla T). \tag{1}$$

Our medium has heat capacity (per mass) of c, and density ρ.

Consider a volume in the medium. The rate of change of the heat Q inside can be written two ways, as a volume integral of a partial in time, and as a flux integral across a surface:

$$\frac{dQ}{dr} = \int_V c\rho \frac{\partial T}{\partial t} dV = -\oint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}}, \tag{2}$$

$$= \oint_S (K\nabla T) \cdot d\vec{\mathbf{A}}, \tag{3}$$

$$= \int_V \nabla \cdot (K\nabla T) dV \quad \text{by Gauss's theore} \tag{4}$$

This has to hold true for any volume V, so we can equate the integrands:

$$c\rho \frac{\partial T}{\partial t} = \nabla \cdot (K\nabla T). \tag{5}$$

If K is a constant,

$$\nabla \cdot (\nabla T) = \nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{6}$$

with $\kappa = K/c\rho$, and the operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{7}$$

is called the Laplacian.

The diffusion equation

The same procedure can be applied to diffusion of particles, if $n(\vec{\mathbf{r}}, t)$ is the concentration of particles per unit volume, the flux is

$$\vec{\mathbf{F}} = -C\nabla \cdot n, \tag{8}$$

where C is a constant. The same sort of analysis as above leads to

$$\nabla^2 n = \frac{1}{C}\frac{\partial n}{\partial t}. \tag{9}$$

If we consider either type of system in a steady state (no time dependence) we get

$$\nabla^2 T = 0 \text{ or } \nabla^2 n = 0, \tag{10}$$

which are both examples of *Laplace's equation*, where

$$\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \tag{11}$$

and just drop the last term for 2–D problems. Remember that in 2–D the boundary is a curve, in 3–D, a surface.

Harmonic functions A harmonic function is a twice continuously differentiable function that satisfies Laplace's equation:

$$\nabla^2 f = 0, \tag{12}$$

inside some boundary. You also sometimes see

$$\Delta f = 0,$$

where Δ is used for ∇^2. Other classic physical examples are the electrical and gravitational (static) potentials outside of charges and masses, respectively.

Harmonic functions have the important property:

> A function that satisfies $\nabla^2 = 0$ cannot have an extremum; the function can only have a maximum or minimum at the edge of the domain on which it is defined.