Ch. 10: Curvature

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Consider a function $f(x)$ that has an extremum. Pick our origin of coordinates at the extremum just to simplify the expressions.

Expand in a Taylor series

$$f(x) = f(0) + \frac{df}{dx}x + \frac{1}{2}\frac{d^2f}{dx^2}x^2 + \cdots \, . \tag{1}$$

The derivatives are evaluated at $x = 0$. This point is a maximum or minimum, so $df/dx = 0$ there. Thus

$$f(x) = f(0) + \frac{1}{2}\frac{d^2f}{dx^2}x^2 + \cdots \tag{2}$$

near the extremum.

Stationary points

A better term for the behavior of the function in this region is that it is a *stationary point,* because in the analogous situation in higher dimensions it is where the derivatives with respect to all variables vanish.

The dominant behavior of the function near its stationary point is given by its curvature, the x^2 term. This can be characterized by a tangent circle whose radius R is the *radius of curvature* of the function there.

The radius of curvature is

$$R = \frac{-1}{f''} \tag{3}$$

and recall the maximum and minimum properties of second derivatives:

$$\text{for a minimum } \frac{\partial^2 f}{\partial x^2} > 0, \tag{4}$$

$$\text{for a maximum } \frac{\partial^2 f}{\partial x^2} < 0. \tag{5}$$

The first is called *concave*, the second *convex.*

Consider a 1–D system with a potential $V(x)$:

$$F(x) = -\frac{dV}{dx} \tag{6}$$

at a stationary point the force vanishes, and this is an equilibrium point. Not all equilibria are the same, however:

$$\frac{\partial^2 V}{\partial x^2} > 0, \text{stable}, \tag{7}$$

$$\frac{\partial^2 V}{\partial x^2} < 0, \text{unstable}. \tag{8}$$

Homework

Chapter 9.2 a–e: get Stoke's theorem from Gauss's theorem in 2–D.

9.5 b–e: The Aharonov-Bohm effect.

Ch. 10 b,c,d: Curvature (should be quick, calculus review.)