Theorem of Stokes

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Start from the idea that the curl is the closed line integral of the field per unit area (see section 7.1).

For an infinitesimal surface element:

$$\oint_{\mathrm{d}\vec{\mathbf{S}}} \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{r}} = (\nabla \times \vec{\mathbf{v}}) \cdot \hat{\mathbf{n}} \mathrm{d}S = (\nabla \times \vec{\mathbf{v}}) \cdot \mathrm{d}\vec{\mathbf{S}}. \quad (1)$$

Here the z-axis is *not* necessarily aligned with $(\nabla \times \vec{\mathbf{v}})$.

This can be integrated over a finite surface S bounded by a curve C:

$$\oint_C \vec{\mathbf{v}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_S (\nabla \times \vec{\mathbf{v}}) \cdot \mathrm{d}\vec{\mathbf{S}} \quad (2)$$

This is Stoke's theorem (or law). This tells you how to compute the integral of the curl of a vector field.

In Stokes law the sense of the line integration and the direction of the surface vector $\mathrm{d}\vec{\mathbf{S}}$ are related through the right-hand rule.

Once you pick the curve C, the surface integral is *independent* of the choice of surface S, as long as S is bounded by C.

This seems a bit strange, but is often useful. Pick the easiest surface to use for a given C.

Magnetic field of a long straight wire

$$\vec{\mathbf{B}} = B(r)\hat{\phi} \tag{3}$$

The field equation that we have to integrate is

$$\nabla \times B = \mu_0 \vec{\mathbf{J}} \tag{4}$$

For just the current through the wire inside our boundary

$$I = \int \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}}.$$

Now we are going to use Stoke's theorem. The left side is

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}}.$$

we are going to go in a circle of radius r about the wire in a plane perpendicular to it. Let the wire lie in cylindrical coords along the z axis. Then $d\vec{\mathbf{r}} = \hat{\phi} r d\phi$

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = (B\hat{\phi}) \cdot (\hat{\phi} r d\phi) = Brd\phi.$$

So the path integral is just

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{r}} = \int_0^{2\pi} Brd\phi = 2\pi rB. \qquad (5)$$

The right side of Stoke's law is

$$\int_S (\nabla \times \vec{\mathbf{B}}) \cdot d\vec{\mathbf{S}} = \int_S (\mu_0 \vec{\mathbf{J}}) \cdot d\vec{\mathbf{S}} \tag{6}$$

The only current density is I through the wire. It moves perpendicular to the wire just like $d\vec{\mathbf{S}}$ points, so this is just

$$= \mu_0 I \tag{7}$$

as noted above. Putting our pieces together

$$2\pi r B(r) = \mu_0 I \tag{8}$$

$$B(r) = \frac{\mu_0 I}{2\pi r} \tag{9}$$

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r}\hat{\phi}. \tag{10}$$

Lenz's Law

Consider a loop of wire in a changing $\vec{\mathbf{B}}$ field. This causes magnetic induction, producing an $\vec{\mathbf{E}}$ field in the wire. See fig. 9.6.

We have the field equation

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}. \tag{11}$$

Integrate over the surface enclosed by the wire:

$$\int_S (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{S}} = -\int_S \frac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{S}} \tag{12}$$

The right side is the time derivative of the flux Φ (see 6.1 for flux).

Work is done per charge as it moves around from A to B.

$$F_{AB} = \int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}, \tag{13}$$

this has the horrid but historical name of "electromotive force." by Stoke's law:

$$\int_S (\nabla \times \vec{\mathbf{E}}) \cdot d\vec{\mathbf{S}} = \int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{r}}. \tag{14}$$

So

$$F_{AB} = -\frac{\partial \Phi_B}{\partial t}. \tag{15}$$

A changing magnetic flux produces an emf in a wire, and the field producing this is such as to oppose the changing magnetic field.