Probability currents

Math Methods, D. Craig

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In quantum mechanics, the wavefunction $\psi(\vec{\mathbf{r}}, t)$ that describes a particle moving in a potential $V(\vec{\mathbf{r}})$ obeys the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

We have not discussed the Laplacian ∇^2 yet, but all we need to know now is that $\nabla^2\psi = \nabla \cdot \nabla\psi$.

The wavefunction tells how *likely* it is the particle will appear at $\vec{\mathbf{r}}$. $|\psi(\vec{\mathbf{r}}, t)|^2$ is the probability density of finding the particle at $(\vec{\mathbf{r}}, t)$. The probability the particle is within a volume V is:

$$P_V = \int_V |\psi|^2 \, dV$$

We'll use these ideas to find the flow of probablity through a surface, similarly to the acoustic representation example.

The wave function is a complex function, for a specific $(\vec{\mathbf{r}}, t)$, $\psi(\vec{\mathbf{r}}, t)$ is a complex number. When we invert the sign of the imaginary part, this is complex conjugation. The complex conjugate of ψ is denoted ψ^*. We will need $\partial\psi^*/\partial t$, so we start by getting the conjugate of

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{1}$$

The complex conjugate $i \to -i$ is

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* \tag{2}$$

(The potential V is a real function.)

The squared magnitude of ψ represents the probability density, and we are going to use

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}(\psi\psi^*) = \psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t} \tag{3}$$

to see how it changes in time. This is why we needed to get $\partial\psi^*/\partial t$ in the previous slide.

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi + V\psi\frac{(-i)}{\hbar} \tag{4}$$

$$\frac{\partial\psi^*}{\partial t} = -\frac{i\hbar}{2m}\nabla^2\psi^* + V\psi^*\frac{(i)}{\hbar} \tag{5}$$

The terms involving V on the right sides cancel when we write out the derivative:

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{i\hbar}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) \tag{6}$$

Now take the volume integral:

$$\frac{\partial}{\partial t}\int_V |\psi|^2 \, dV = \frac{i\hbar}{2m}\int_V (\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) \, dV \tag{7}$$

using the fact that $\nabla^2\psi = \nabla \cdot \nabla\psi$ and the identity (8.9) in the text:

$$\psi^*\nabla^2\psi = \nabla \cdot (\psi\nabla\psi^*) - \nabla\psi^* \cdot \nabla\psi, \tag{8}$$

$$\psi\nabla^2\psi^* = \nabla \cdot (\psi^*\nabla\psi) - \nabla\psi \cdot \nabla\psi^*. \tag{9}$$

When we put these in the integral above, we are left with

$$\frac{\partial}{\partial t}\int_V |\psi|^2 \, dV = \frac{i\hbar}{2m}\int_V [\nabla \cdot (\psi\nabla\psi^*) - \nabla \cdot (\psi^*\nabla\psi)] \, dV, \tag{10}$$

the volume integral of divergences.

As before, by Gauss's Law this becomes a surface integral:

$$\frac{\partial}{\partial t}\int_V |\psi|^2 \, dV = \frac{i\hbar}{2m}\oint_S (\psi^* \nabla \psi - \psi \nabla \psi^*) \cdot d\vec{\mathbf{S}}. \quad (11)$$

The LHS above is the time derivative of the probability the particle is within V. The RHS describes the "flow" of probability through the surface S. This can be written

$$\frac{\partial P_V}{\partial t} = -\int \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}}, \quad (12)$$

with the probability density current

$$\vec{\mathbf{J}} = \frac{i\hbar}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi). \quad (13)$$