Divergence: radial solutions, sec. 6.2,6.5 Cross product review

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To understand the orbit stability example (section 6.5) one detail that is needed is very similar to Problem 6.2(a): Using

$$r = \sqrt{x^2 + y^2}$$

Show that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \tag{1}$$

and then that

$$(\nabla \cdot \vec{\mathbf{v}}) = 2f(r) + r\frac{df}{dr} \tag{2}$$

when the field is radially symmetric:

$$\vec{\mathbf{v}} = f(r)\vec{\mathbf{r}}. \tag{3}$$

The first part is simple:

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2)^{1/2} \tag{4}$$

$$= 2x(\frac{1}{2})(x^2 + y^2)^{-1/2} \tag{5}$$

$$= \frac{x}{(x^2 + y^2)^{1/2}} = \frac{x}{r} \tag{6}$$

For the next step, the key is to realize that simply

$$\vec{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}.$$

Forgetting this type of simple approach can drive you batty in vector calculus.

So, to take the divergence:

$$(\nabla \cdot \vec{\mathbf{v}}) = \frac{\partial}{\partial x}(f(r)x) + \frac{\partial}{\partial y}(f(r)y), \tag{7}$$

$$= x\frac{\partial}{\partial x}f(r) + f(r)\frac{\partial x}{\partial x}, + y\frac{\partial}{\partial y}f(r) + f(r)\frac{\partial y}{\partial y}, \tag{8}$$

$$= x\frac{df}{dr}\frac{\partial r}{\partial x} + f(r) + y\frac{df}{dr}\frac{\partial r}{\partial y} + f(r), \tag{9}$$

using our previous result:

$$= 2f(r) + \left(\frac{x^2 + y^2}{r}\right)\frac{df}{dr}, \tag{10}$$

$$= 2f(r) + r\frac{df}{dr}, \text{ Q.E.D} \tag{11}$$

The same result extended to N dimensions is **Problem 6.5 b.**

Quick review of cross product.

It is an inherently three-dimensional product*

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta \hat{\mathbf{n}} \tag{12}$$

θ is the angle between $\vec{\mathbf{a}}, \vec{\mathbf{b}}$. $\hat{\mathbf{n}}$ is normal to the plane containing them. The orientation of $\hat{\mathbf{n}}$ is chosen by the *handedness* of the coordinate system, usually right-hand-rule nowadays. This dependence on the choice for handedness makes it a *psuedovector.*

The magnitude

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta$$

is the area of the parallelogram spanned by $\vec{\mathbf{a}}, \vec{\mathbf{b}}$.

The fact that it can be used to get areas and normals is very important in computer graphics, particularly in 3-d rendering systems.

*Apparently it may also be defined consistently in 7 dimensions. See Wikipedia.

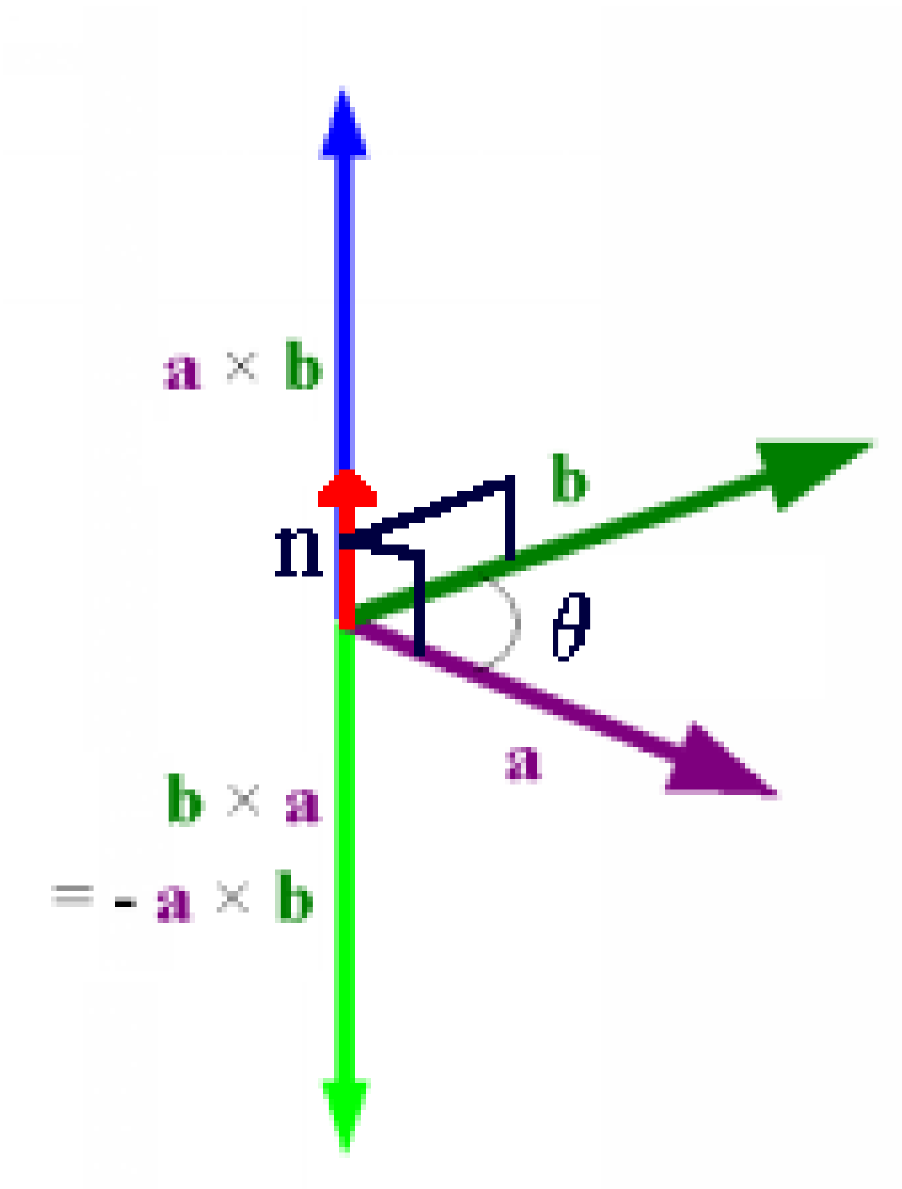

Image from

http://en.wikipedia.org/wiki/Cross_product

Algebraic properties

Cross product is **anticommutative,** and is in general **not associative.** Hence many identities using it are non-obvious. Keep references at hand when doing vector algebra.

Calculation

Four ways to remember it:

- Symbolic determinant
- Skew-symmetric matrix
- Levi-Civita symbol and indices
- Brute memorization (ick.)

Let $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3 = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, then

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{13}$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk} \hat{\mathbf{e}}_i a_j b_k \tag{14}$$

The **Levi-Civita symbol** ϵ_{ijk} is

$$\epsilon_{ijk} = \begin{cases} +1 & \text{when } (i,j,k) = (1,2,3),(2,3,1),(3,1,2); \\ -1 & \text{when } (i,j,k) = (3,2,1),(2,1,3),(1,3,2); \\ 0 & \text{all other cases (any repeated indices).} \end{cases}$$

Note the cyclic order—this is how you can remember it: if the indices go forward from 1, it is +1 (wrap at end), otherwise -1, 0 if any repeats.

There is an interesting graphical representation of ϵ_{ijk} at Wikipedia:

`http://en.wikipedia.org/wiki/Levi_civita_symbol`

The skew-symmetric matrix form:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \mathbf{A}_{\times}\mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}. \quad (15)$$