A 5-d world? Stability of orbits in higher dimensions

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We are going to consider gravitational fields in a space of dimension N using the divergence to calculate the field strength and hence the force.

(This is actually also using a gravitational version of Gauss's Law)

For a source of gravity field:

$$(\nabla \cdot \vec{\mathbf{g}}) = -4\pi G\rho$$

Gravitational field is spherically symmetric

$$\vec{\mathbf{g}}(\vec{\mathbf{r}}) = f(r)\vec{\mathbf{r}}$$

In N dimensions, the definition of distance from a point is

$$r = \sqrt{\sum_{i=1}^{N} x_i^2}$$

which leads to

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{r}$$

for each x_j.

From this we can get the divergence of $f(r)\vec{\mathbf{r}}$ in N dimensions (prob 6.5b)

$$(\nabla \cdot \vec{\mathbf{g}}) = Nf(r) + r\frac{\partial f}{\partial r}.$$

Outside the sun the divergence vanishes, and this can be integrated as a d. e. to get

$$\vec{\mathbf{g}}(\vec{\mathbf{r}}) = -\frac{A}{r^{N-1}}\hat{\mathbf{r}}$$

A doesn't matter for our stability argument.

$$\vec{\mathbf{F}}_{\text{grav}} = -\frac{Am}{r^{N-1}}\hat{\mathbf{r}}$$

If we consider ourselves to be in a rotating reference frame* following the planet in a circular orbit, there is a centrifugal force

$$F_{\text{cent}} = \frac{mv^2}{r}\hat{\mathbf{r}}.$$

And these must balance for the circular orbit to stay in equilibrium:

$$\vec{\mathbf{F}}_{\text{grav}} + \vec{\mathbf{F}}_{\text{cent}} = 0.$$

This gives us a speed (prob. 6.5 d)

$$v = \sqrt{\frac{A}{r^{N-2}}}$$

for a circular orbit.

*Remember that the centrifugal force is a psuedoforce produced by an accelerated frame.

Perturbation

Now assume something wiggles our planet:

$$r \to r + \delta r, \quad \delta\vec{\mathbf{r}} = \delta r \hat{\mathbf{r}}.$$

The forces will also change in this case.

First assume $\delta r > 0$, we move away from the Sun, $\delta\vec{\mathbf{r}}$ points outward. The orbit will be stable if the change in the forces points *back*to the Sun:

$$(\delta\vec{\mathbf{F}}_{\text{grav}} + \delta\vec{\mathbf{F}}_{\text{cent}}) \cdot \delta\vec{\mathbf{r}} < 0 \text{ stability.}$$

If we move *in* $\delta\vec{\mathbf{r}}$ points toward the sun, so we want the perturbation of forces to point *away*, which means the above is still true: this is our criterion for stability.

In order to get the change in centrifugal force, use conservation of angular momentum:

$$mrv = m(r + \delta r)(v + \delta v)$$
$$rv = rv + r\delta v + v\delta r + \delta r\delta v$$

drop the second order terms, cancel rv

$$0 = r\delta v + v\delta r$$
$$\delta v = -\frac{v}{r}\delta r$$

To show stability, we can use, for example

$$\delta F_{\text{grav}} = \frac{\partial F_{\text{grav}}}{\partial r}\delta r$$

to get

$$\delta \vec{\mathbf{F}}_{\text{cent}} = -\frac{3mv^2}{r^2}\delta\vec{\mathbf{r}}$$
$$\delta \vec{\mathbf{F}}_{\text{grav}} = (N-1)\frac{Am}{r^N}\delta\vec{\mathbf{r}}.$$

These can be substituted into the stability condition to find that orbits are stable in less than four dimensions, as shown on the next slide.

$$\left(-\frac{3mv^2}{r^2}\delta\vec{\mathbf{r}} + (N-1)\frac{Am}{r^N}\delta\vec{\mathbf{r}}\right)\cdot\delta\vec{\mathbf{r}} < 0$$

The dot products of the vectors will be positive, so

$$\left(-\frac{3mv^2}{r^2} + (N-1)\frac{Am}{r^N}\right) < 0.$$

Substitute in expression for v^2:

$$\begin{aligned}\left(-\frac{3mA}{r^2(r^{N-2})} + (N-1)\frac{Am}{r^N}\right) &< 0,\\ \left(-\frac{3}{r^N} + \frac{N-1}{r^N}\right) &< 0,\\ -3 + N - 1 &< 0\\ N &< 4.\end{aligned}$$

So stability under small radial perturbations requires $N < 4$, so a gravitational field will only have stable orbits in three dimensions or less.