Newton’s Second from a potential

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Suppose we have a particle moving in a system characterized by a potential $V(\vec{\mathbf{r}})$. Its total energy is the sum of the kinetic energy and potential energy:

$$\frac{1}{2}mv^2 + V(\vec{\mathbf{r}}) = E.$$

Assume that E is conserved, so

$$\frac{dE}{dt} = 0.$$

We want to take the time derivative of both sides of the first equation. First term is not too hard:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}mv^2\right) &= \frac{m}{2}\frac{d}{dt}(v_x^2 + v_y^2 + v_z^2) \\
&= \frac{m}{2}\left(2v_x\frac{dv_x}{dt} + 2v_y\frac{dv_y}{dt} + 2v_z\frac{dv_z}{dt}\right) \\
&= m\left(\vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{v}}}{dt}\right).
\end{aligned}$$

For the time derivative of the potential energy, we have to remember that even though V itself doesn't change in time, $\vec{\mathbf{r}}(t)$ does. So we have

$$\frac{dV(\vec{\mathbf{r}})}{dt} = \lim_{\delta t \to 0} \frac{V(\vec{\mathbf{r}}(t+\delta t)) - V(\vec{\mathbf{r}}(t))}{\delta t},$$

with $\delta V = V(\vec{\mathbf{r}}(t+\delta t)) - V(\vec{\mathbf{r}}(t))$. Now $\delta V = \nabla V \cdot \delta \vec{\mathbf{r}}$ so

$$\frac{dV(\vec{\mathbf{r}})}{dt} = \lim_{\delta t \to 0} \frac{\nabla V \cdot \delta r}{\delta t} = (\vec{\mathbf{v}} \cdot \nabla V).$$

Putting these results together:

$$\vec{\mathbf{v}} \cdot \left(m \frac{d\vec{\mathbf{v}}}{dt} + \nabla V \right) = 0.$$

$\vec{\mathbf{v}}$ is any arbitrary velocity, so the term in parentheses must always be zero:

$$m \frac{d\vec{\mathbf{v}}}{dt} = -\nabla V = \vec{\mathbf{F}},$$

which is Newton's second law.

So Newton's second law can be derived from energy conservation. We have also shown that

$$\vec{\mathbf{F}} = -\nabla V,$$

which is an important general idea: forces arise from changes in the potential energy function acting on a particle.

In two or three dimensions the change along a path can be characterized by the gradient.