Ch. 5: Gradient 1

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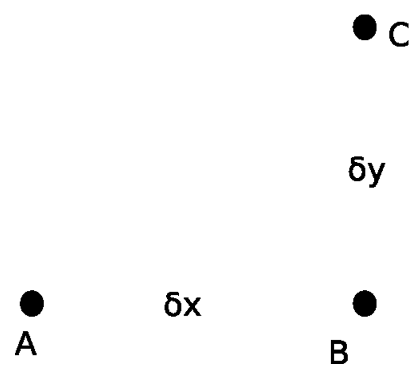
Gradient of a function



Consider a point A at (x, y), B at $(x + \delta x, y)$, and C at $(x + \delta x, y + \delta y)$, with values of f at these points.

$$\begin{aligned} \delta f &= f_C - f_A, \\ &= f_B + (f_C - f_A) - f_B, \\ &= f_B - f_A + f_C - f_B. \end{aligned}$$

Then these can be written

$$f_B - f_A = \frac{\partial f}{\partial x}(x, y)\, \delta x$$
$$f_C - f_B = \frac{\partial f}{\partial y}(x + \delta x, y)\, \delta y$$

So the change in the function value f is

$$\delta f = \frac{\partial f}{\partial x}(x, y)\, \delta x + \underbrace{\frac{\partial f}{\partial y}(x + \delta x, y)\, \delta y}_{\text{note}}$$

$$\frac{\partial f}{\partial y}(x + \delta x, y)\, \delta y \to \frac{\partial f}{\partial y}(x, y)\, \delta y$$

because from Taylor expansion:

$$\frac{\partial f}{\partial y}(x + \delta x, y)\, \delta y = \frac{\partial f}{\partial y}(x, y)\, \delta y + \frac{\partial^2 f}{\partial x \partial y}(x, y)\, \underbrace{\delta x \delta y}_{\text{vanish}}$$

So we have

$$\delta f = \frac{\partial f}{\partial x}(x, y)\, \delta x + \frac{\partial f}{\partial y}(x, y)\delta y.$$

Recall $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y$:

$$\delta \vec{\mathbf{r}} = (\vec{\mathbf{r}}_C - \vec{\mathbf{r}}_A)$$
$$\delta \vec{\mathbf{r}} = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

and define a (column) vector

$$\nabla f = \begin{pmatrix} \partial f / \partial x \\ \partial f / \partial y \end{pmatrix}$$

So a small change in f is given by

$$\delta f = (\nabla f \cdot \delta \vec{\mathbf{r}}),$$

where ∇f is a vector * called the **gradient**. Also sometimes written grad x.

*There is a more modern vector space approach to this. It is a *dual vector* or *1-form*.

If

$$\delta f = (\nabla f \cdot \delta\vec{\mathbf{r}}),$$

from the definition of dot product

$$\delta f = |\nabla f||\delta\vec{\mathbf{r}}| \cos\theta.$$

θ is the angle between the gradient and $\delta\vec{\mathbf{r}}$.

∇f points toward the direction of increasing f. If $\delta\vec{\mathbf{r}}$ points along the gradient, then $\theta = 0$, and

$$\delta f = |\nabla f||\delta\vec{\mathbf{r}}|,$$
$$|\nabla f| = \frac{\delta f}{|\delta\vec{\mathbf{r}}|}$$

In terms of contour lines, the gradient points "uphill", and the closer the contour lines, the steeper the gradient. The gradient will have dimensions of whatever f is divided by length.

Three dimensions

This generalizes directly to three dimensions:

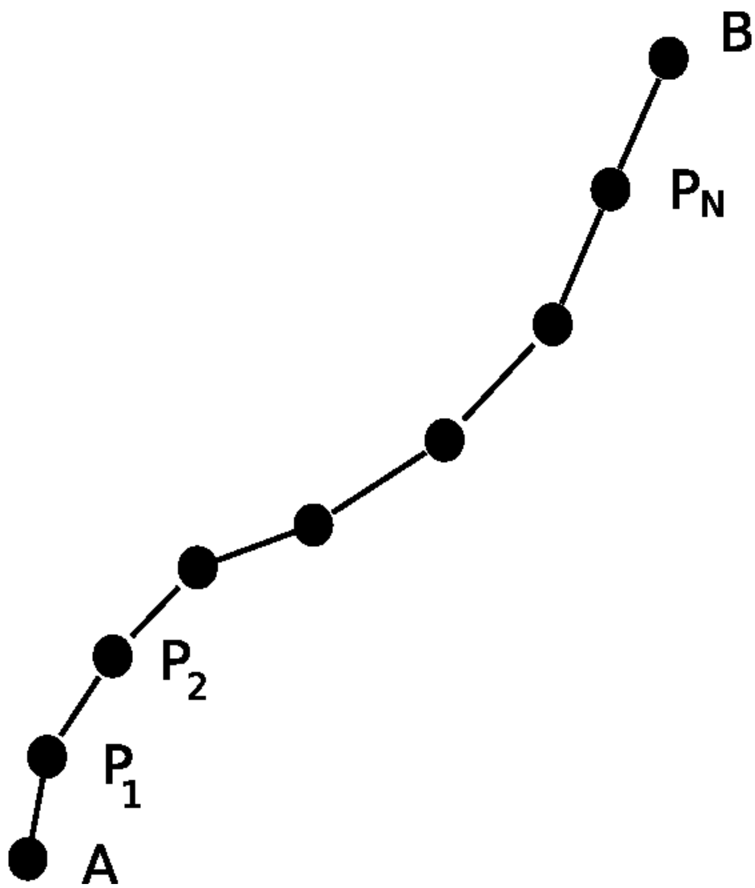
$$\delta\vec{\mathbf{r}} = \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$$

Beyond 3-D, more machinery is needed, and the modern approach there uses tensor calculus, and/or multivectors or vectors and dual vectors.

Integration

In one-dimensional calculus, a change over a finite interval can be seen as a sum of infinitesimal steps, which leads to the definition of the definite integral. In two or three dimensions we have a change over a *path* instead.



The change in the function value from A to B will be

$$f_B - f_A = (f_B - f_N) + (f_N - f_{N-1}) + \cdots + (f_2 - f_1) + (f_1 - f_A)$$

each step has a δf associated

$$\delta f = (\nabla f \cdot \delta \vec{\mathbf{r}})$$

with $\delta \vec{\mathbf{r}}$'s along (tangent to) the path. So

$$f_B - f_A = \sum_N (\nabla f \cdot \delta \vec{\mathbf{r}}).$$

Let $N \to \infty$ and $\delta \vec{\mathbf{r}} \to \mathbf{d}\vec{\mathbf{r}}$ to get

$$f_B - f_A = \int_A^B (\nabla f \cdot \mathbf{d}\vec{\mathbf{r}}),$$

where the integral is along *any* path from A to B.

If you close the path, you come back to the same point, so

$$\oint (\nabla f \cdot \mathbf{d}\vec{\mathbf{r}}) = 0.$$

The path integral of the gradient along a closed path vanishes.

Differentiation: directional derivative

Define a derivative in a direction $\hat{\mathbf{n}}$.

$$\frac{df}{ds}(\vec{\mathbf{r}}) = \lim_{\delta s \to 0} \frac{f(\vec{\mathbf{r}} + \hat{\mathbf{n}}\delta s) - f(\vec{\mathbf{r}})}{\delta s} = \frac{\delta f}{\delta s}$$

with $\delta f = (\nabla f \cdot \delta\vec{\mathbf{r}})$. In this case $\delta r = \hat{\mathbf{n}}\delta s$ and so

$$\begin{aligned}\frac{df}{ds}(\vec{\mathbf{r}}) &= \frac{(\nabla f \cdot \hat{\mathbf{n}})\delta s}{\delta s} \\ &= (\hat{\mathbf{n}} \cdot \nabla f).\end{aligned}$$

This is the derivative of f in the direction $\hat{\mathbf{n}}$.

So we have generalized the derivative to a path integral, and the derivative to a directional derivative, in moving to 2 and 3 dimensional space.