Spherical and Cylindrical Coordinates 1

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Many problems exhibit spherical or cylindrical symmetry, at least approximately.

Whenever possible **use a coordinate system that fits the symmetry of the problem.** This almost always simplifies calculations.

Spherical coordinates

r is the radius from the origin of the point.

ϕ is the angle in the xy plane of the projection of the point "down" onto that plane, measured from the $+x$ axis. It runs from $0 \to 2\pi$. This is much easier to see in a diagram.

θ is the angle from the $+z$ axis of the line from the origin to the point.

See figure 4.1.

Beware! Various texts may have different conventions for which one is θ, which is ϕ. Always look for a diagram.

Remember we can represent the position vector

$$\vec{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

where $\hat{\mathbf{x}}$ means unit vector, length 1. Any arbitrary vector can be written

$$\vec{\mathbf{u}} = u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}} + u_z\hat{\mathbf{z}}.$$

Relating $x, y, z \leftrightarrow r, \theta, \phi$

From the geometry of the diagram (picture worth at least 1 kiloword):

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

This is problem 4.1b, and takes careful trig reasoning from the diagram. You can invert these to get relations:

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arccos(z/r)$$
$$\phi = \arctan(y/x).$$

A careful look at the diagram helps here too, 4.1c.

Relating unit vectors We would also like to be able to write

$$\vec{\mathbf{u}} = u_r \hat{\mathbf{r}} + u_\theta \widehat{\theta} + u_\phi \widehat{\phi},$$

so we want to relate the coefficients

$$(u_x, u_y, u_z) \overset{?}{\leftrightarrow} (u_r, u_\theta, u_\phi).$$

The key: $\hat{\mathbf{x}}$ points along the x-axis. So it is unit vector **pointing in the direction of increasing x for constant** y, z.

We can say that:

$$\hat{\mathbf{x}} = \frac{\partial \vec{\mathbf{r}}}{\partial x}.$$

Show this (4.1d):

$$\vec{\mathbf{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\hat{\mathbf{x}} = \frac{\partial}{\partial x} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now what about an arbitrary unit vector? Think about $\hat{\theta}$:

- $\hat{\theta}$ points toward increasing θ for r, ϕ constant,

- so $\hat{\theta} = C(\partial\vec{\mathbf{r}}/\partial\theta)$,

- where C is such to make $\hat{\theta}$ unit length.

Which leads to (4.1e)

$$\hat{\mathbf{r}} = \frac{\partial \vec{\mathbf{r}}}{\partial r}, \quad \hat{\theta} = \frac{1}{r}\frac{\partial \vec{\mathbf{r}}}{\partial \theta}, \quad \hat{\phi} = \frac{1}{r\sin\theta}\frac{\partial \vec{\mathbf{r}}}{\partial \phi}.$$

and if you take the partials these can be written as column vectors:

$$\hat{\mathbf{r}} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix},$$

$$\hat{\theta} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}$$

$$\hat{\phi} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

You can show that the unit vectors form an orthogonal set by taking dot products in this form:

Example from Problem 4.1f:

$$
\begin{aligned}
\hat{\theta} \cdot \hat{\theta} &= \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \\
&= \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta \\
&= \cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta \\
&= \cos^2\theta + \sin^2\theta \\
&= 1
\end{aligned}
$$