Dealing with small variations

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2007–01–26

Section 3.2: Cosmic dust

The example in this section shows a technique using the first order term of the Taylor series to make a numerical estimate.

I will go through the solution with comments. This technique for dealing with small variations is very useful for quick estimates, and also shows some important considerations for numerical calculations.

3.2 a:

5×10^7 kg/a of meteoric dust falls on Earth. What is the volume per year (a for annum in the units) ? For average meteoric material $\rho = 2.5 \times 10^3$ kg/m^3. So

$$\delta V = \frac{m}{\rho} = 2 \times 10^4 \text{ m}^3.$$

We call it δV because this will be change in Earth's volume per year.

The relation between radius and volume $V = (4\pi/3)r^3$ can be written:

$$r = \left(\frac{3V}{4\pi}\right)^{1/3}.$$

So the change in radius is:

$$\delta r = \left[\frac{3(V + \delta V)}{4\pi}\right]^{1/3} - \left(\frac{3V}{4\pi}\right)^{1/3}.$$

Now we say "Ha! We're done!" Reach for the calculator. . .

3.2 Problem b

$$
\begin{aligned}
V &= 1.437 \times 10^{21}\ \text{m}^3 \text{ is the volume of Earth.} \\
V + \delta V &= 1.436755040 \times 10^{21}\ \text{m}^3 + 2 \times 10^4\ \text{m}^3 \\
&= 1.436755040 \times 10^{21}\ \text{m}^3 \text{ Uh oh } \ldots
\end{aligned}
$$

To something like 17 digits, both terms in the expression for δr are going to be the same. So your calculator will say $\delta r = 0$.

You **may** get away with the right answer if your calculator has an algebra system or a many-digit mode, such as a TI89/92, or if you use a computer algebra system such as Maple or Mathematica. But that is swatting a fly with a sledgehammer.

3.2 Problem c

Look back at (3.18):

$$f(x+h) - f(x) \approx h\frac{df}{dx}(x).$$

Just identify $h \to \delta x$ and $f(x+h) - f(x) \to \delta f$. Then it's obvious:

$$\delta f \approx \frac{df}{dx}\delta x$$

The text has a partial derivative, it's same as the derivative for this one-dimensional case. (May need to check general case.)

3.2 Problem d Getting the derivative in the right form is nontrivial. Write r like this:

$$r(V) = \left(\frac{3}{4\pi}\right)^{1/3} \cdot V^{1/3}$$

to get the constant out of the way. Then take the derivative:

$$\frac{dr(V)}{dV} = \left(\frac{3}{4\pi}\right)^{1/3} \cdot \frac{1}{3} V^{-2/3}$$

$$= \frac{1}{3}\left(\frac{3}{4\pi}\right)^{1/3} \cdot V^{(\frac{1}{3}-1)}$$

$$= \frac{1}{3}\left[\left(\frac{3}{4\pi}\right)^{1/3} V^{1/3}\right] V^{-1}$$

The thing in brackets is r again, so:

$$= \frac{1}{3}\frac{r}{V}$$

So we arrive at the result:

$$\delta r = \frac{1}{3} r \frac{\delta V}{V}.$$

and plugging in the values we see that the Earth grows 1 angstrom per year from meteorites.