2.4: Lift of a wing

Math Methods, D. Craig

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Buckingham pi theorem

If a problem contains N variables that depend on P physical dimensions, then there are $N-P$ dimensionless numbers that describe the physics of the problem.

The previous and following examples show the basic procedure for finding such a dimensionless number, at least for simple cases.

In level steady flight, the lift of a wing must equal the weight of the aircraft or bird, so denote lift by W.

$$W \sim [\mathrm{F}] \sim [\mathrm{ML/T^2}].$$

The lift depends on the mass density ρ of the air, the flow velocity v, and the surface area S of the wing.

We have 4 variables and 3 physical dimensions, so the by the Buckingham pi theorem there is $N - P = 1$ dimensionless number characteristic of this problem.

The dimensions are

$$\rho \sim [M/L^3],$$
$$v \sim [L/T],$$
$$S \sim [L^2].$$

We want to express W in terms of the others, so

$$W\rho^{\alpha}\nu^{\beta}S^{\gamma} \sim [1].$$

Expanding:

$$[MLT^{-2}][ML^{-3}]^{\alpha}[LT^{-1}]^{\beta}[L^2]^{\gamma} = C^0$$
$$M^1L^1T^{-2}M^{\alpha}L^{-3\alpha}L^{\beta}T^{-\beta}L^{2\gamma} = M^0L^0T^0$$

Leads to

$$1+\alpha = 0,$$
$$1-3\alpha+\beta+2\gamma = 0,$$
$$2+\beta = 0.$$

This has solution

$$\alpha = \gamma = -1, \ \beta = -2.$$

Then we put this back in:

$$W\rho^{-1}v^{-2}S^{-1} = C_L$$

So, the expression for W is

$$W = C_L \rho v^2 S.$$

C_L is called the *lift coefficient.* It depends on the wing shape and the angle of attack (angle with the airflow).

Using the steps of section 2.5 this can be used to show that the lift scales as v^6 for a given C_L, and the scatter about the line in fig. 2.2 is due to varying C_L for different aircraft and animals.