Derivation of optical dispersion for a single type of atomic oscillator.

D. Craig, WTAMU

February 3, 2011

This is a rewriting with more explicit steps and different commentary of the derivation in Hecht's *Optics,* section 3.5.

Imagine a plane electromagnetic wave moving through a medium consisting of a bunch of polarizable atoms. The relationship between the perimitivity ϵ, the electric field, and the polarization $\vec{P}$ is

$$(\epsilon - \epsilon_0)\vec{E} = \vec{P}. \tag{1}$$

$\vec{P}$ is the dipole moment per unit volume. We will use this to figure out $\epsilon(\omega)$, then using the Maxwell relation $n^2 = K_E = \epsilon/\epsilon_0$ we obtain a frequency-dependent $n(\omega)$.

We can think of each electron cloud and atom as a system that the electric field will drive in oscillation. The force exerted by the electric field on the electron is

$$F_E = q_e E(t) = q_e E_0 \cos \omega t. \tag{2}$$

We are using a harmonic wave at a frequency ω as the "forcing" field for the oscillation. Now the electron cloud will have a restoring force, whose precise force law is complex, but remember that any restoring force law for small displacements will be approximately like Hooke's law. So the restoring force will be

$$F = -k_E x \tag{3}$$

where k_E is an "elastic" constant of electrical (and quantum) origin. But we know this will have a resonant frequency, so write it in terms of that resonance ω_0:

$$\omega_0 = \sqrt{\frac{k_E}{m_e}}, \tag{4}$$

$$k_E = m_e \omega_0^2, \tag{5}$$

which gives a restoring force

$$F = -m_e \omega_0^2 x, \tag{6}$$

where x is the displacement from equilibrium. We hence have two forces on the electron (or electron "cloud") in a given atom: the driving force from the electric field and the restoring force due to the electron orbital. What to do next? Write Newton's Second Law![1]

$$q_e E_0 \cos \omega t - m_e \omega_0^2 x = m_e \frac{d^2 x}{dt^2}. \tag{7}$$

While this is the second law, it is in the form of a second-order, inhomogeneous differential equation. What to do? Two things: use physical reasoning, and guess.[2] We suspect that the electron will oscillate at the same frequency, and so we can write a trial solution

$$x(t) = x_0 \cos \omega t. \tag{8}$$

Substitute this solution into the d.e. and solve for the unknown amplitude x_0:

$$q_e E_0 \cos \omega t - m_e \omega_0^2 x_0 \cos \omega t = -m_e \omega^2 x_0 \cos \omega t, \tag{9}$$

$$q_e E_0 - m_e \omega_0^2 x_0 = -m_e \omega^2 x_0, \tag{10}$$

and after rearranging

$$x_0 = \frac{q_e/m_e}{(\omega_0^2 - \omega^2)} E_0. \tag{11}$$

Substitute this back into our expression for $x(t)$:

$$x(t) = \frac{q_e/m_e}{(\omega_0^2 - \omega^2)} E_0 \cos \omega t. \tag{12}$$

[1] It is amazing how often one does this, even in advanced work.

[2] This is the route to solving many d.e.'s.

Now we have an expression for the displacement of the electron cloud from equilibrium and this will produce a polarization in the material, which will modify the dielectric constant, and thus the index of refraction. For a plane wave we can essentially use one-dimensional expressions. The polarization per volme will be

$$P = q_e x N, \tag{13}$$

where N is the number of polarizable atoms per volume, the density of induced dipoles caused by the imposed electric field. Thus

$$P = \frac{q_e^2 N E / m_e}{(\omega_0^2 - \omega^2)}. \tag{14}$$

Now use the relation between P and ϵ:

$$(\epsilon - \epsilon_0) E = P$$

to get

$$\epsilon = \epsilon_0 + \frac{P(t)}{E(t)} = \epsilon_0 + \frac{q_e^2 N / m_e}{(\omega_0^2 - \omega^2)}, \tag{15}$$

and then Maxwell's relation $n^2 = K_E = \epsilon/\epsilon_0$, to get

$$n^2(\omega) = 1 + \frac{N q_e^2}{\epsilon_0 m_e} \left(\frac{1}{\omega_0^2 - \omega^2} \right). \tag{16}$$