Harmonic waves, 1-D propagation

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1 Moving waves, function arguments

Figure 2.3 in Hecht is an excellent explanation of why an argument of $(x - vt)$ to a function gives an unchanging wave shape propagating in the positive direction. Study it carefully to understand this conceptually.

Any function of the form $\psi(x, t)$ whose two arguments can be written as one variable $x' = x - vt$ will be such a wave. If you choose some function that varies in space $f(x')$ to give a wave shape, then the moving wave will be

$$\psi = f(x') = f(x - vt). \tag{1}$$

1.1 General case

$$\psi = f(x \mp vt), \tag{2}$$
$$\psi = f(x - vt), \text{ (traveling positive direction)} \tag{3}$$
$$\psi = f(x + vt). \text{ (traveling negative direction)} \tag{4}$$

Note well that f doesn't have to be a sine or cosine, but sines and cosines are important because

- they result from linear, lossless oscillation phenomena (in an ideal case),
- they have clearly defined power, etc. (for appropriate physical systems),
- any other function can be written as a sum of sine and cosine terms.

The last idea is particulary important in physical optics. Note that as a function, a sine wave persists forever, and this is unphysical. But it is very useful to be able to decompose a physical function into sinusoids because of their nice mathematical properties. Also, a sinusoid is a good approximation to a physical wave which lasts for many cycles.

2 Differential wave equation

2.1 Mathematical form

Get this under your skin. It and its variations are everywhere in physics, especially modern physics. The simplest form is

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}, \tag{5}$$

where ψ is any "wave quantity", which in optics is usually the electric field strength or some related quantity.

This is the form without sources, or damping. These add extra terms, etc., but this is the form for propagation of a lossless wave in one dimension.

2.2 How to remember it consistently

Use dimensional analysis: you have to relate ψ to itself involving the second partials in time and space. The dimensions of the partials will be

$$\frac{[\psi]}{[L]^2} = (\text{something}) \frac{[\psi]}{[T]^2}, \tag{6}$$

where $[L], [T]$ are the length and time dimensions, and $[\psi]$ is whatever the dimensions of ψ are. The only thing (something) can be to make the dimensions consistent is $1/[L/T]^2$, which are the dimensions of $1/v^2$:

$$\frac{[\psi]}{[L]^2} = \frac{1}{[L/T]^2} \frac{[\psi]}{[T]^2}. \tag{7}$$

So now you know where to put v^2, and get:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}, \tag{8}$$

This idea will allow you to remember the form of the wave equation, and if you use v^2 instead of $1/v^2$, you'll put it on the proper side. Paying attention to dimensions (a.k.a units) works even in more advanced physics!

3 Harmonic waves

Write a harmonic wave as

$$\psi(x, t) = A \sin k(x - vt) = f(x - vt), \tag{9}$$

where k is the propagation number, or wavenumber, A is the amplitude.

3.1 Harmonic wave quantities

We will also need to deal with the following quantities:

wavelength: λ, $\psi(x,t) = \psi(x \pm \lambda, t)$, the 'spatial period';

period: τ, the time for the wave to repeat at a point in space;

frequency: ν,or often f, the number of cycles per time at a point in space;

angular frequency: ω, the rate at which the phase is advancing in rad/s.

For reasons that are unclear to me, ν is popular with many authors for frequency. Be sure to distinguish it from v. I may slip into using f instead.

3.2 Things that make 2π

A bunch of these multiply to 2π:

$$k\lambda = kv\tau = \omega\tau = 2\pi, \tag{10}$$

which are useful for relating them. Other important simple relations are

$$\nu = 1/\tau, \tag{11}$$

$$v = \nu\lambda. \tag{12}$$

3.3 Most common forms, and phase

The most common forms of a harmonic travelling wave that starts at 0 at $t, x = 0$ are:

$$\psi = A \sin k(x \mp vt), \tag{13}$$

$$\psi = A \sin(kx \mp \omega t). \tag{14}$$

For many other variations, see the equations at the bottom of page 16.

The overall argument, $(kx \mp \omega t)$ is often called the 'phase'. For a wave with some other value at the initial time and position, moving in the positive direction, we can write:

$$\psi(x,t) = A \sin(kx - \omega t + \epsilon), \tag{15}$$

where ϵ is the **initial phase.**

4 Phase velocity

Write the phase

$$\phi(x,t) = (kx - \omega t + \epsilon) \tag{16}$$

The rate of change of phase with time is

$$\left| \left(\frac{\partial \phi}{\partial t} \right)_x \right| = \omega. \tag{17}$$

The rate of change of phase with distance is

$$\left| \left(\frac{\partial \phi}{\partial x} \right)_t \right| = k. \tag{18}$$

A particular phase value corresponds to a point on the profile of the sine, so the speed of propagation of the condition of constant phase is

$$\left(\frac{\partial x}{\partial t} \right)_\phi = \frac{-(\partial \phi / \partial t)_x}{(\partial \phi / \partial x)_t}. \tag{19}$$

(This comes from the properties of partial derivatives.) Plugging these in,

$$\left(\frac{\partial x}{\partial t} \right)_\phi = \pm \frac{\omega}{k} = \pm v. \tag{20}$$

This is the speed at which the sinusoidal profile moves, the *phase speed*. It is important to note that this is not necessarily the same as the signal or group velocity, which is the speed at which information or energy can be transmitted (more later).