Superposition (2.4), Complex Representation (2.5)

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The Wave Equation:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

Is a homogenous PDE, contains no terms in independent variables: no "source" terms. This holds outside the "generator" region of the waves.

The Superposition Principle

If wave functions ψ_1 and ψ_2 are solutions of the wave equation, then $(\psi_1 + \psi_2)$ is also a solution.

This is easy to show by adding corresponding sides of the wave equations for the two solutions.

Look at the two solutions separately

$$\frac{\partial^2\psi_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi_1}{\partial t^2}\ ;\ \frac{\partial^2\psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi_2}{\partial t^2}$$

add them:

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi_1}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2\psi_2}{\partial t^2}$$

and so

$$\frac{\partial^2(\psi_1+\psi_2)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2(\psi_1+\psi_2)}{\partial t^2}$$

hence $(\psi_1 + \psi_2)$ is a solution.

These are *linear* waves: any multiple of a solution is a solution and any sum of solutions is a solution.

For linear superposition, when waves overlap the **resulting disturbance at each point is the algebraic sum of the individual constituent waves at that location.**

(see figures 2.13, 2.14)

Most of the phenomena in Optics are consequences!

Complex representation, or, practical uses of complex numbers

$$\tilde{z} = x + iy$$
$$i = \sqrt{-1}$$

We can write in a polar form (fig 2.15a)

$$x = r\cos\theta$$

and

$$y = r\sin\theta$$

so

$$\tilde{z} = x + iy = r(\cos\theta + i\sin\theta)$$

The *Euler formula*

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

lets you write

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

and back to

$$\tilde{z} = re^{i\theta} = r\cos\theta + ir\sin\theta.$$

r is the *magnitude* of $\tilde(z)$;θ is the *phase angle* of $\tilde{z}$. Magnitude is also written $|\tilde{z}|$.

Any complex number can be written as a sum of real and imaginary parts:

$$\tilde{z} = \mathrm{Re}(\tilde{z}) + \mathrm{Im}(\tilde{z}).$$

and for example

$$\mathrm{Re}(\tilde{z}) = r\cos\theta.$$

We can represent a harmonic wave as

$$\psi(x,t) = \mathrm{Re}\left[Ae^{i(\omega t - kx + \epsilon)}\right] = \mathrm{Re}[Ae^{i\phi}]$$

which is equivalent to

$$\psi(x,t) = A\cos(\omega t - kx + \epsilon).$$