Something shocking about traffic

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1 Introduction

The dynamics of traffic flow exhibits a host of nonlinear phenomena, such as kinematic and dynamic shock waves. The following treatment follows Chapter 22 of *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics,* by Robert B. Banks, 1998, Princeton University Press.

There are demos at:

http://www.webs1.uidaho.edu/niatt_labmanual/Chapters/trafficflowtheory/theoryandconcepts/ShockWaves.htm

and

http://www.ece.osu.edu/~coifman/shock/

Amazing applets at:

http://rcswww.urz.tu-dresden.de/~helbing/RoadApplet/

http://www.mtreiber.de/MicroApplet

2 Mathematics of traffic flow

Consider a particular stretch of highway, with vehicles entering from the left and exiting on the right, say between two boundary marks a distance dx apart.

The rate at which vehicles (particles) cross a boundary is Q, the *flux rate* in vehicles/hr. In general $Q = Q(x,t)$. At left hand boundary, flux is Q, at right hand boundary, dx further along, the flux is

$$Q(x + dx) = Q(x) + \frac{\partial Q}{\partial x} dx. \tag{1}$$

The density of vehicles is K vehicles/mile, and in general $K = K(x,t)$. The number of vehicles in a stretch dx will be $K\ dx$. The rate of change of vehicles within that dx will be $\partial(K\ dx)/\partial t$. These "acccumulate" inside dx.

Now we conserve cars (mass):

$$\text{(cars entering)} - \text{(cars leaving)} = \text{cars accumulating.} \tag{2}$$

Symbolically this is

$$Q - \left(Q + \frac{\partial Q}{\partial x} dx\right) = \frac{\partial K}{\partial t}\, dx. \tag{3}$$

One can easily see this simplifies to

$$\frac{\partial Q}{\partial x} + \frac{\partial K}{\partial t} = 0. \tag{4}$$

This is known as the *equation of continuity,* which is really just a statement of conservation of mass, particles, or vehicles. Note that there are *two independent* variables, x and t, and two *dependent* variables, $Q(x, t)$ and $K(x, t)$.

Now let's introduce the *flow velocity*, or average speed of the vehicles.

$$U \frac{\text{mi}}{\text{hr}} = \frac{Q(\text{vehicles/hr})}{K(\text{vehicles/mi})}. \tag{5}$$

This tells us

$$Q = UK, \tag{6}$$

that is, flow Q is product of velocity and concentration.

People tend to slow down as the traffic density increases. Let's model this with a simple linear relationship (see figure 1):

$$U = U_* \left(1 - \frac{K}{K_*}\right). \tag{7}$$

U_* is the maximum v, and K_* is the jam concentration, the density at which traffic will stop ("bumper-to-bumper".)

This gives us

$$Q = U_* K \left(1 - \frac{K}{K_*}\right), \tag{8}$$

a simple parabolic relationship between Q and K (figure 2). If we set $\frac{dQ}{dK} = 0$ to find the maximum flow Q_m, we find that $Q_m = U_* K_*/4$ which occurs when $K_m = K_*/2$. The relationship $Q(K)$ given by equation (8) is an *equation of state* for the system, like an equation of state for a gas.

For single-lane traffic, some typical numbers are

$$Q_m = 2000 \text{ vehicles/hr} \tag{9}$$

$$K_* = 160 \text{ vehicles/mi} \tag{10}$$

This gives

$$K_m = 80 \text{ vehicles/mi, and} \tag{11}$$

$$U_* = 50 \text{ mi/hr maximum velocity.} \tag{12}$$

The mean traffic velocity (for maximum flux of vehicles) will be

$$U_m = U_*/2 = 25 \text{ mi/hr}. \tag{13}$$

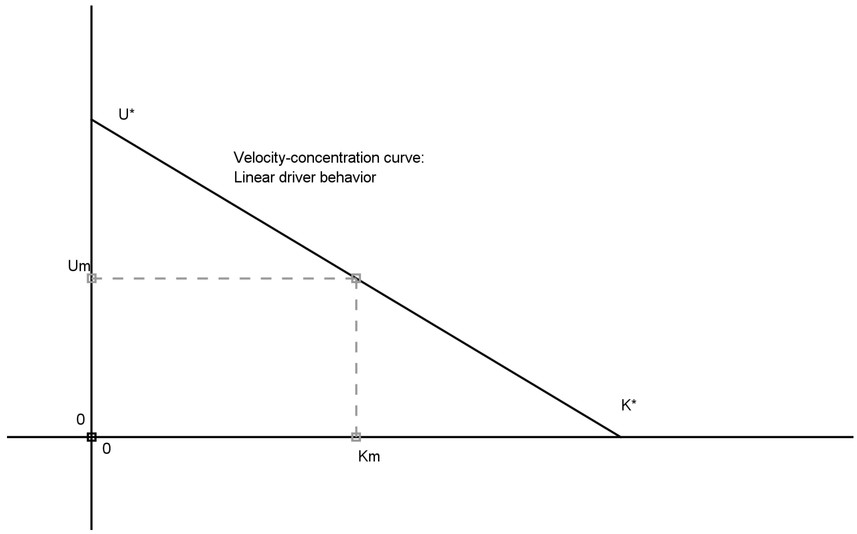

Figure 1: Velocity vs. concentration for a linear "driver behavior".

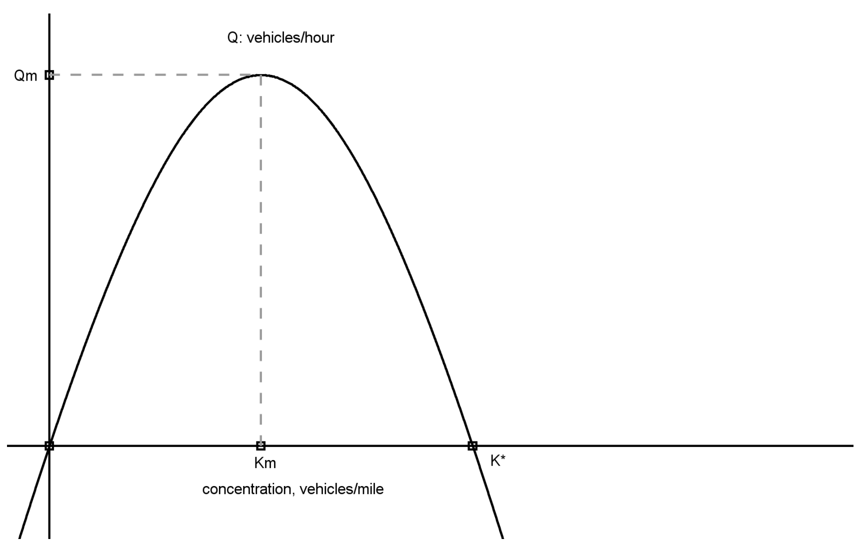

Figure 2: Vehicle flux Q versus concentration.

3 Kinematic waves

We can rewrite the continuity equation

$$\frac{\partial K}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{14}$$

$$\frac{\partial K}{\partial t} + \frac{dQ}{dK}\frac{\partial K}{\partial x} = 0 \tag{15}$$

which can be put in the form

$$\frac{\partial K}{\partial t} + C\frac{\partial K}{\partial x} = 0 \tag{16}$$

Where $C = dQ/dK$ is the velocity of a kinematic wave moving through the vehicle stream, given by our equation of state. Differentiating, we get

$$C = \frac{dQ}{dK} = U_* \left(1 - \frac{2k}{K_*}\right). \tag{17}$$

We also have from our other way of writing Q, equation (6):

$$C = \frac{dQ}{dK} = \frac{d(UK)}{dK} = U + K\frac{dU}{dK}. \tag{18}$$

Note that $dU/dK < 0$, so always $C < U$ so kinematic waves are slower than the flow velocity,or moving in the opposite direction.[1]

Exercise. Sketch U versus K and Q versus K and figure out which slopes are the vehicle and wave velocities.

If we look at the graph of capacity Q versus concentration K, it's a parabola. On the left of K_m, the road is carrying fewer cars than its maximum flow.The slope of a line from the origin to a point on the parabola is $V = Q/K$, the average vehicle velocity. The slope of the *tangent* to the parabola is the kinematic wave velocity, $C = dQ/dK$. If $K < K_m$, then $C > 0$ and a little less than V, so the "wave" in this case travels a little slower than traffic in the same direction. This is like when traffic is speeding up from a slowdown: the cars spread out at as they move away and the speed of the "spreading" group of cars is slightly less than the cars.

For $K > K_m$, as marked in figure 3, the road is overfilled: the density is in excess of the density for the maximum flow of vehicles. A tangent to the parabola will have a *negative* slope, so $C < 0$, which means the wave propagates against the flow of traffic. If traffic is grinding to a halt as $K \to K_*$, then $V \to 0$, and the wave moves rapidly *backwards:* the cars are stacking up as fast as they come in.

[1] There was a typo in the earlier version of this article, Equation (18) is correct.

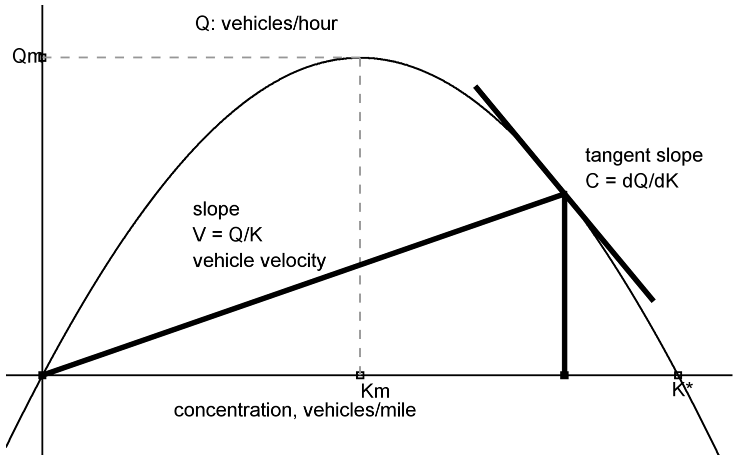

Figure 3: Traffic flow diagram. The slope from origin to a point on the parabola is the vehicle velocity, the tangent to the parabola at that point is the shock velocity.

4 Kinematic waves and characteristics

Now look back at equation (16). This is a *nonlinear first-order partial differential equation.* It has the general solution

$$K = f(x - Ct)$$

Where f is a function given by the initial conditions. This will propagate without changing form at velocity C. The concentration K is a function of the quantity $(x - Ct)$, where C depends on the particular K, as discussed above.

If we plot the position of the wave in the $(x - t)$ plane, we get a family of linescalled the *characteristics.* The slope of each characteristic is given by the local wave velocity $C = dQ/dK$. Along each characteristic, K is constant and equal to its initial $(t = 0)$ value.

A solution to eq. (16) is given by a family of characteristic lines in the $(x-t)$ plane, each corresponding to a value of K from the initial conditions.

See

`http://en.wikipedia.org/wiki/Method_of_characteristics`

and links there.

Traffic waves are called *kinematic* waves because they are mathematically generated without bringing in any force relationships. Kinematic waves are due to relatively gentle adjustments of driver speeds. When kinematic waves collide, shock waves are generated.

5 Shock waves

When two sets of characteristic lines intersect, i.e. where two kinematic waves meet, a shock wave is formed. Its velocity is

$$C_* = \frac{Q_2 - Q_1}{K_2 - K_1}$$

where the numbers represent the differing regions of flow. This will be a chord across the traffic parabola between the two points representing conditions on each side.

(to be continued.)