Jackson 9.5 Homework Problem Solution

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PROBLEM:

(a) Show that for harmonic time variation at frequency *ω* the electric dipole scalar and vector potentials in the Lorenz gauge and the long-wavelength limit are

$$\Phi(\mathbf{x})=\frac{e^{ikr}}{4\pi\epsilon_0 r^2}\mathbf{n}\cdot\mathbf{p}(1-ikr)$$

$$\mathbf{A}(\mathbf{x})=-i\frac{\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\mathbf{p} \quad \text{[this is (9.16)]}$$

where $k = \omega/c$, $\boldsymbol{n}$ is a unit vector in the radial direction, **p** is the dipole moment (9.17), and the time dependence $e^{-i\omega t}$ is understood.

(b) Calculate the electric and magnetic fields *from the potentials* and show that they are given by (9.18).

SOLUTION:

(a) Define potentials according to $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$ and Maxwell's equations in free space are reduced to the following equations:

$$\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J}-\frac{1}{c^2}\nabla\frac{\partial}{dt}\Phi-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} \quad \text{and} \quad \nabla^2\Phi+\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A})=-\frac{\rho}{\epsilon_0}$$

In the Lorenz gauge, $\nabla\cdot\mathbf{A}=-\frac{1}{c^2}\frac{\partial\Phi}{\partial t}$, these equations become wave equations:

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J} \quad \text{and} \quad \nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=-\frac{\rho}{\epsilon_0}$$

As demonstrated in class, the solution to this type of differential equation (wave equation with sources) can be found using the Green function method. The final solution for radiation is an integral of static solutions, retarded to ensure causality:

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}\left(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)d\mathbf{x}' \quad , \quad \Phi(\mathbf{x},t)=\frac{1}{4\pi\epsilon_0}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho\left(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)d\mathbf{x}'$$

For a harmonic time variation in frequency:

$$\mathbf{A}(\mathbf{x},t)=\mathbf{A}(\mathbf{x})e^{-i\omega t} \quad , \quad \mathbf{J}(\mathbf{x},t)=\mathbf{J}(\mathbf{x})e^{-i\omega t} \quad \text{, and} \quad \Phi(\mathbf{x},t)=\Phi(\mathbf{x})e^{-i\omega t}$$

The solutions become:

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' \quad , \quad \Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')e^{ik|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

In the long-wavelength limit, we apply $|\mathbf{x}-\mathbf{x}'|=r-r'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'$ and $1/|\mathbf{x}-\mathbf{x}'|=(1+(r'/r)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')/r$ to find:

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int(1+(r'/r)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')\mathbf{J}(\mathbf{x}')e^{-ikr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'}d\mathbf{x}' \quad ,$$

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int(1+(r'/r)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')\rho(\mathbf{x}')e^{-ikr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'}d\mathbf{x}'$$

Expand the exponential in the integral according to $e^x=1+x+\frac{1}{2}x^2+\ldots$

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[\int(1+(r'/r)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')\mathbf{J}(\mathbf{x}')d\mathbf{x}'-ik\int(1+(r'/r)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')\mathbf{J}(\mathbf{x}')(r'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')d\mathbf{x}'+\ldots\right] \quad ,$$

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\int(1+(r'/r)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')\rho(\mathbf{x}')d\mathbf{x}'-ik\int(1+(r'/r)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')\rho(\mathbf{x}')(r'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')d\mathbf{x}'+\ldots\right]$$

Keep only the electric dipole terms:

$$\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\mathbf{J}(\mathbf{x}')d\mathbf{x}' \quad ,$$

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r^2}\mathbf{n}\cdot\mathbf{p}(1-ikr) \quad \text{where} \quad \mathbf{p}=\int\mathbf{x}'(\rho(\mathbf{x}'))d\mathbf{x}'$$

Perform an integration by part on the first integral:

$$\mathbf{A}(\mathbf{x},t)=-\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\mathbf{x}'(\nabla'\cdot\mathbf{J}(\mathbf{x}'))d\mathbf{x}'$$

From the continuity equation for harmonic time dependence, we know $\nabla'\cdot\mathbf{J}(\mathbf{x}')=i\omega\rho(\mathbf{x}')$ so that finally:

$$\boxed{\mathbf{A}(\mathbf{x},t)=-\frac{i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{p}}$$

$$\boxed{\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r^2}\mathbf{n}\cdot\mathbf{p}(1-ikr)} \quad \text{where} \quad \mathbf{p}=\int\mathbf{x}'(\rho(\mathbf{x}'))d\mathbf{x}'$$

Note that we have made no far-field approximation, only a large wavelength approximation. These equations are valid for all zones.

(b) We can now calculate the electric and magnetic fields from the potentials.

$$\mathbf{H}=\frac{1}{\mu_0}\nabla\times\mathbf{A}$$

$$\mathbf{H}=\nabla\times\left(-\frac{i\omega}{4\pi}\frac{e^{ikr}}{r}\mathbf{p}\right)$$

Use the identity: $\nabla\times[\mathbf{p}\,f(r)]=(\mathbf{n}\times\mathbf{p})\frac{\partial f(r)}{\partial r}$

$$\mathbf{H}=-\frac{i\omega}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)$$

$$\mathbf{H}=-\frac{i\omega}{4\pi}(\mathbf{n}\times\mathbf{p})\left(\frac{-e^{ikr}}{r^2}+ik\frac{e^{ikr}}{r}\right)$$

$$\boxed{\mathbf{H}=\frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)}$$

The electric field is:

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{E}=-\nabla\left(\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r^2}\mathbf{n}\cdot\mathbf{p}(1-ikr)\right)-\frac{\partial}{\partial t}\left(-\frac{i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{p}\right)$$

$$\mathbf{E}=-\frac{1}{4\pi\epsilon_0}\left(\nabla\left(\frac{e^{ikr}}{r^2}\mathbf{n}\cdot\mathbf{p}\right)-ik\nabla\left(\frac{e^{ikr}}{r}\mathbf{n}\cdot\mathbf{p}\right)\right)+\frac{\omega^2\mu_0}{4\pi}\frac{e^{ikr}}{r}\mathbf{p}$$

Use $\nabla(f(r)\mathbf{n}\cdot\mathbf{A})=\frac{f(r)}{r}\mathbf{A}+(\mathbf{n}\cdot\mathbf{A})\mathbf{n}\left(\frac{\partial f(r)}{\partial r}-\frac{f(r)}{r}\right)$ to find:

$$\boxed{\mathbf{E}=\frac{e^{ikr}}{4\pi\epsilon_0}\left(\frac{k^2}{r}(\mathbf{n}\times\mathbf{p})\times\mathbf{n}+(3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p})\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\right)}$$