Jackson 9.3 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell

PROBLEM:

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V \cos \omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

SOLUTION:

Two halves of a sphere that are charged oppositely should immediately remind us of a dipole so that we can assume the dipole term is the dominant term. We just need to find the dipole moment of this configuration and then we can apply the equations in the class notes for the fields radiated by a dipole.

Let us consider the sphere at an instant in time when the voltages are at their peak. We previously found the external potential due to such a sphere in terms of Legendre polynomials. We found the first term to be:

$$\Phi = V \frac{3}{2} \frac{R^2}{r^2} \cos\theta$$

We know that the potential due to a electric dipole pointing in the z direction should look like:

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{p}{r^2} \cos\theta$$

Setting these equal and solving for p we end up with

$$\boxed{\mathbf{p} = 6\pi\epsilon_0 V R^2 \hat{\mathbf{z}}}$$

Now the potentials on the sphere vary according to cos(ωt), which is just the real part of the complex exponential signifying harmonic dependence. We already found in the class notes the radiation-zone fields due to an electric dipole varying harmonically in time. We can write them down immediately and plug in the dipole moment we have here.

$$\mathbf{B} = \frac{\mu_0 c k^2 p}{4\pi} (\hat{\mathbf{k}} \times \hat{\mathbf{p}}) \frac{e^{i(kr - \omega t)}}{r}$$

$$\boxed{\mathbf{B} = -\frac{3}{2} \frac{V k^2 R^2}{c} \frac{e^{i(kr - \omega t)}}{r} \sin\theta \, \hat{\boldsymbol{\phi}}}$$

$$\mathbf{E}=-\frac{k^2 p}{4\pi\epsilon_0}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{i(kr-\omega t)}}{r}$$

$$\boxed{\mathbf{E}=-\frac{3}{2}V k^2 R^2\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\boldsymbol{\theta}}}$$

The time-averaged angular distribution of radiated power is:

$$\frac{dP}{d\Omega}=\frac{1}{2}\Re[r^2\hat{\mathbf{r}}\cdot\mathbf{E}\times\mathbf{H}^*]$$

$$\frac{dP}{d\Omega}=\frac{1}{2}\Re[r^2\hat{\mathbf{r}}\cdot\left[-\frac{3}{2}V k^2 R^2\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\boldsymbol{\theta}}\right]\times\left[-\frac{1}{\mu_0}\frac{3}{2}\frac{V k^2 R^2}{c}\frac{e^{-i(kr-\omega t)}}{r}\sin\theta\,\hat{\boldsymbol{\phi}}\right]]$$

$$\boxed{\frac{dP}{d\Omega}=\frac{9}{8}\frac{V^2 k^4 R^4}{\mu_0 c}\sin^2\theta}$$

The total radiated power is:

$$P=\int\frac{9}{8}\frac{V^2 k^4 R^4}{\mu_0 c}\sin^2\theta\, d\Omega \quad , \quad \boxed{P=\frac{3\pi V^2 k^4 R^4}{\mu_0 c}}$$