Jackson 8.5 Homework Problem Solution

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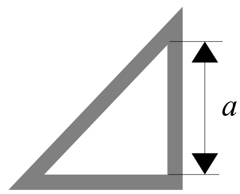
PROBLEM:

(a) A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a, a, $\sqrt{2}a$, as shown. The medium inside has $\mu_r = \varepsilon_r = 1$. Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.



SOLUTION:

(a) If we tried to solve the waveguide outright, we would soon discover that the triangle geometry does not let us choose a natural coordinate system for the boundary conditions in which we can use separation of variables. Fortunately, this problem has enough symmetry that we can use a trick. The right triangle waveguide can be thought as a special case of the square waveguide with an extra boundary condition along the diagonal $y = x$.

For TE modes, the square waveguide had solutions:

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)e^{ikz - i\omega t} \quad \text{where } m \text{ and } n = 0, 1, 2, \ldots \text{ and } \quad \kappa^2 = \frac{\pi^2}{a^2}(m^2+n^2)$$

A quick calculation shows these already satisfy the boundary conditions on the bottom and right walls:

$$\left[\frac{\partial B_Z}{\partial n}\right]_S = 0$$

The solution to the triangle waveguide can be a constructed as a superposition of the square waveguide solutions. But which ones? Whichever ones satisfy:

$$\left[\frac{\partial B_Z}{\partial n}\right]_{y=x} = 0$$

$$\frac{1}{\sqrt{2}}\left[\frac{\partial B_Z}{\partial x} - \frac{\partial B_Z}{\partial y}\right]_{y=x} = 0$$

$$\left[\frac{\partial B_Z}{\partial x} = \frac{\partial B_Z}{\partial y}\right]_{y=x}$$

This can be satisfied automatically by constructing:

$$B_z = B_z^{\text{sq}}(x, y) + B_z^{\text{sq}}(y, x)$$

$$\boxed{B_z = B_0 \left[\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) \right] e^{ikz - i\omega t}}$$

where $m \geq n \geq 0$ must hold but $m = n = 0$ is not allowed.

For TM modes, the solution to the square waveguide is:

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) e^{ikz - i\omega t} \quad \text{where } m \text{ and } n = 1, 2, \ldots \text{ and } \quad \kappa^2 = \frac{\pi^2}{a^2}(m^2 + n^2)$$

A quick calculation shows these already satisfy the boundary conditions on the bottom and right walls:

$$\left[E_z\right]_S = 0$$

The solution to the triangle waveguide can be a constructed as a superposition of the square waveguide solutions. We must satisfy:

$$\left[E_z\right]_{x=y} = 0$$

It should be obvious that this is automatically satisfied if we construct:

$$E_z = E_z^{\text{sq}}(x, y) - E_z^{\text{sq}}(y, x)$$

because when $x = y$, the two terms become identical and their sum vanishes.

$$\boxed{E_z = E_0 \left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) - \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right) \right] e^{ikz - i\omega t}} \quad \text{where } m > n > 0$$

The cutoff frequencies are the same as for the square waveguide:

$$\boxed{\omega_{mn} = \frac{c\pi}{a}\sqrt{m^2 + n^2}}$$