Jackson 8.4 Homework Problem Solution

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PROBLEM:

Transverse electric and magnetic waves are propagated along a hollow, right, circular cylinder with inner radius R and conductivity σ.

(a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part, assume that the conductivity of the cylinder is infinite.

SOLUTION:

(a) In a waveguide, the transverse and axial components of the fields become coupled together so that we only have to solve for one set of components and then can use the waveguide equations to find the other components. We can choose whether to solve for the transverse or axial components. For this problem, it will probably be easier to solve for the axial components.

For TE modes, there is no axial electric field, so we are solving for B_z. There are no charges, currents, or materials in the waveguide, so the field obey the homogeneous wave equation:

$$\nabla^2 B_z - \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2} = 0$$

The wave is free along the axis of the waveguide (the z axis), so we can assume the modes have simple harmonic dependence in this direction and are harmonic in time $B_z(x,y,z,t) = B_z(x,y)e^{ikz-i\omega t}$. Putting this form in the wave equation, we find:

$$\nabla_t^2 B_z = -\kappa^2 B_z \quad \text{where} \quad \kappa^2 = \frac{\omega^2}{c^2} - k^2$$

The Laplacian operator here is only in the transverse directions. Because of the circular symmetry, we express the Laplacian in polar coordinates:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial B_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 B_z}{\partial \phi^2} = -\kappa^2 B_z$$

Try a solution of the form $B_z = R(\kappa\rho)e^{im\phi}$ where the single-value requirement (because the whole span of azimuth angles is included in the region of interest) forces m to be an integer.

$$\rho^2 \frac{\partial^2 (R(\kappa\rho))}{\partial \rho^2} + \rho \frac{\partial (R(\kappa\rho))}{\partial \rho} + (\kappa^2\rho^2 - m^2) R(\kappa\rho) = 0$$

Make a change of variables, $x = \kappa\rho$

$$x^2 \frac{\partial^2 R(x)}{\partial x^2} + x \frac{\partial R(x)}{\partial x} + \left(x^2 - m^2\right) R(x) = 0$$

This is the Bessel differential equation. The solution to this equation involves Bessel and Neumann functions:

$$R(x) = A J_m(x) + B N_m(x)$$

The solution for a particular mode is therefore:

$$B_z = \left(A J_m(\kappa\rho) + B N_m(\kappa\rho)\right) e^{i(kz - \omega t + m\phi)}$$

We now apply boundary conditions. The boundary condition for the minimum radius is simply that the solution must be finite along the axis of the waveguide. This forces us to drop the Neumann functions which blow up at zero radius.

$$B_z = A J_m(\kappa\rho) e^{i(kz - \omega t + m\phi)}$$

At the surface of the perfect conductor constituting the walls of the waveguide, we have the boundary condition:

$$\left[\frac{\partial B_z}{\partial \rho}\right]_{\rho = R} = 0$$

Applying this boundary condition:

$$\left[\frac{\partial}{\partial \rho}\left(J_m(\kappa\rho)\right)\right]_{\rho = R} = 0$$

The zeros of this equation are found numerically and published in lists. Let us call them x'_{mn}:

$$\kappa = \frac{x'_{mn}}{R}$$ where x'_{mn} is the n^{th} zero of $J'_m(x)$.

Our particular solution for the TE mode m, n is therefore:

$$\boxed{B_z = A_{mn} J_m\left(x'_{mn} \frac{\rho}{R}\right) e^{i(kz - \omega t + m\phi)}}$$ where $k = \sqrt{\frac{\omega^2}{c^2} - \frac{x'^2_{mn}}{R^2}}$ and x'_{mn} is the n^{th} zero of $J'_m(x)$

Let us now solve for the TM case, where we are looking for E_z. The axial electric field obeys the wave equation just like the axial magnetic field, so that we have the same solution to the differential equation:

$$E_z = A J_m(\kappa \rho) e^{i(kz - \omega t + m\phi)} \quad \text{where} \quad k = \sqrt{\frac{\omega^2}{c^2} - k^2}$$

The difference now is that the boundary condition at the walls is $E_z(\rho = R) = 0$. Applying this boundary condition, we have:

$$J_m(\kappa R) = 0$$

We call the zeros of this equation x_{mn}:

$$\kappa = \frac{x_{mn}}{R}$$ where x_{mn} is the n^{th} zero of $J_m(x)$.

Our particular solution for the TM mode m, n is therefore:

$$\boxed{E_z = A_{mn} J_m\left(x_{mn} \frac{\rho}{R}\right) e^{i(kz - \omega t + m\phi)}} \quad \text{where} \quad k = \sqrt{\frac{\omega^2}{c^2} - \frac{x_{mn}^2}{R^2}}$$ and x_{mn} is the n^{th} zero of $J_m(x)$.

The cutoff frequencies are the frequencies where the wavenumber equals zero:

$$\omega_{mn} = c \frac{x'_{mn}}{R} \quad \text{for TE modes}$$

$$\omega_{mn} = c \frac{x_{mn}}{R} \quad \text{for TM modes}$$

The roots of the Bessel functions can be looked up. The first few are listed below. Note that by convention, the first zero is $n = 1$. There is nothing represented by $n = 0$.

x_{mn}	$m = 0$	$m = 1$	$m = 2$
$n = 1$	2.40	3.83	5.14
$n = 2$	5.52	7.02	8.42
$n = 3$	8.65	10.17	11.62

x'_{mn}	$m = 0$	$m = 1$	$m = 2$
$n = 1$	0.00	1.84	3.05
$n = 2$	3.83	5.33	6.71
$n = 3$	7.02	8.54	9.97

Note that the TE_{01} mode, shown in pink, is not a valid mode because $\kappa = 0$, which leads the waveguide equations to explode. Therefore, omitting the TE_{01} mode, the lowest value in these tables is the valie of 1.84 for the TE_{11} mode. This is the fundamental mode. The first five lowest-mode cutoff frequencies are:

$$\mathrm{TE}_{11}: \omega_c = 1.84 \frac{c}{R}$$

$$\mathrm{TM}_{01}: \omega_c = 2.40 \frac{c}{R}$$

$$\mathrm{TE}_{21}: \omega_c = 3.05 \frac{c}{R}$$

$$\mathrm{TE}_{02} \text{ and } \mathrm{TM}_{11}: \omega_c = 3.83 \frac{c}{R}$$

In terms of the dominant frequency, the cutoff frequencies are:

$$\mathrm{TE}_{11}: \omega_{dom}/\omega_{dom} = 1$$

$$\mathrm{TM}_{01}: \omega_c/\omega_{dom} = 1.30$$

$$\mathrm{TE}_{21}: \omega_c/\omega_{dom} = 1.66$$

$$\mathrm{TE}_{02} \text{ and } \mathrm{TM}_{11}: \omega_c/\omega_{dom} = 2.08$$