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Jackson 8.2 Homework Problem Solution

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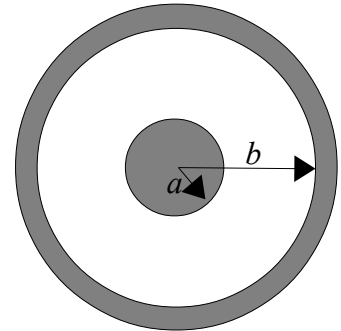
PROBLEM:

A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ , as shown, is filled with uniform lossless dielectric (μ , ϵ). A TEM mode is propagated along this line. Section 8.1 applies.

(a) Show that the time-averaged power flow along the line is $P = \sqrt{\frac{\mu}{\epsilon}} \pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$ where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

(b) Show that the transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z} \quad \text{where} \quad \gamma = \frac{1}{2\sigma\delta} \sqrt{\frac{\epsilon}{\mu}} \frac{\left(\frac{1}{a} + \frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$



(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z . Show that for this line:

$$Z_0 = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln\left(\frac{b}{a}\right)$$

(d) Show that the series resistance and inductance per unit length of the line are:

$$R = \frac{1}{2\pi\sigma\delta} \left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{and} \quad L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi} \left(\frac{1}{a} + \frac{1}{b}\right)$$

where μ_c is the permeability of the conductor. The correction to the inductance comes from the penetration of the flux into the conductors by a distance of order δ .

SOLUTION:

(a) By definition a TEM mode is a single-frequency wave component with both the electric field and magnetic field transverse to the direction of propagation, which is the waveguide's axis. As described in the book, this fact reduces the problem to a two-dimensional pseudo-static problem and ensures the free-space relation $\mathbf{B} = \sqrt{\mu\epsilon} \hat{\mathbf{z}} \times \mathbf{E}$ still holds. It should be clear from the symmetry of this problem and its static nature that the electric fields will point in the radial direction and the magnetic field will point in the azimuthal direction. Additionally, they will both be constant in the azimuthal direction. The inner conductor has some uniform total charge per unit length λ . Draw a cylindrical Gaussian surface around

the inner conductor and apply Gauss' law to find:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon\rho} \hat{\rho}$$

Using the relation above, the magnetic field is therefore:

$$\mathbf{B} = \sqrt{\mu\epsilon} \frac{\lambda}{2\pi\epsilon\rho} \hat{\phi}$$

As directed by the problem description, we set the peak magnetic field at the surface of the inner conductor at a known value H_0 .

$$H(\rho=a) = H_0 = \frac{1}{\sqrt{\mu\epsilon}} \frac{\lambda}{2\pi a}$$

Solving for the total charge per into length, we find $\lambda = 2\pi a H_0 \sqrt{\mu\epsilon}$ so that the fields become:

$$\mathbf{E} = \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{a}{\rho} \hat{\rho} \quad \text{and} \quad \mathbf{B} = \mu H_0 \frac{a}{\rho} \hat{\phi}$$

This is the solution to the static problem. The solution to the waveguide problem is therefore:

$$\mathbf{E} = \sqrt{\frac{\mu}{\epsilon}} H_0(z) \frac{a}{\rho} \hat{\rho} e^{ikz - i\omega t} \quad \text{and} \quad \mathbf{B} = \mu H_0(z) \frac{a}{\rho} \hat{\phi} e^{ikz - i\omega t}$$

Note that we have only the problem so far for a general two-dimensional slice at some location z . In general, H_0 can depend on z , and indeed it must because of the attenuation arising from non-infinite conductivity.

The energy flux is therefore:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\mathbf{S} = \frac{1}{\mu} \left[\sqrt{\frac{\mu}{\epsilon}} |H_0(z)| \frac{a}{\rho} \hat{\rho} \cos(kz - \omega t + \text{Arg}(H_0)) \right] \times \left[\mu |H_0(z)| \frac{a}{\rho} \hat{\phi} \cos(kz - \omega t + \text{Arg}(H_0)) \right]$$

$$\mathbf{S} = \sqrt{\frac{\mu}{\epsilon}} |H_0(z)|^2 \frac{a^2}{\rho^2} \cos^2(kz - \omega t + \text{Arg}(H_0)) \hat{\mathbf{z}}$$

The time-averaged power flux is:

$$\langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} |H_0(z)|^2 \frac{a^2}{\rho^2} \hat{\mathbf{z}}$$

The total power is therefore:

$$P = \int_A \hat{\mathbf{z}} \cdot \langle \mathbf{S} \rangle da$$

$$P = \int_0^{2\pi} \int_a^b \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} |H_0(z)|^2 \frac{a^2}{\rho^2} \rho d\rho d\phi$$

$$P = \sqrt{\frac{\mu}{\epsilon}} \pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$$

Note that the total power flow does not seem to depend on the conductivity or skin depth, which is unexpected. In fact, P does depend on the conductor's properties, but this dependence is hidden with H_0 .

(b) Although power is flowing down the waveguide in the fields, some of the power is also flowing into the conductor and being lost to heat because of the non-infinite conductivity. As a result, the traveling wave dies down as it travels because it loses energy to the conducting material. Within the assumptions of a good conductor as detailed in the book in Section 8.1, the power lost is:

$$\frac{dP}{dz} = \frac{1}{4} \mu_c \omega \delta |\mathbf{H}_{\parallel}|^2 \quad \text{where } \mu_c \text{ is the permeability, and } \delta \text{ the skin depth of the conductor}$$

The area element in this equation is a patch on the conductor's surface. Let us factor out the azimuthal part of this area element and integrate over it explicitly:

$$\frac{dP_{\text{loss}}}{dz} = \frac{1}{4} \mu_c \omega \delta \int_0^{2\pi} |\mathbf{H}_{\parallel}|^2 \rho d\phi$$

There are two boundaries, so that the total power loss is

$$\frac{dP_{\text{loss}}}{dz} = \frac{\pi}{2} \mu_c \omega \delta \left[|\mathbf{H}_{\parallel}(a)|^2 a + |\mathbf{H}_{\parallel}(b)|^2 b \right]$$

We know the magnetic field from part a and can insert it now:

$$\frac{dP_{\text{loss}}}{dz} = \frac{\pi}{\sigma \delta} |H_0|^2 a \left[1 + \frac{a}{b} \right]$$

This is the rate at which energy enters the conductors and gets absorbed and is therefore the negative of the rate at which the power propagating down the waveguide decreases: $\frac{dP}{dz} = -\frac{dP_{\text{loss}}}{dz}$

The more power there is available, the quicker it is lost. Assume a linear relationship between the power available and how fast it's lost, with the proportionality constant being (2γ) :

$$-\frac{dP}{dz} = 2\gamma P$$

The solution to this equation is: $P(z) = P_0 e^{-2\gamma z}$

The guided wave is therefore exponentially decaying due to the power loss. The characteristic loss distance can be found:

$$\gamma = -\frac{1}{2P} \frac{dP}{dz}$$

$$\gamma = \frac{1}{2P} \frac{dP_{\text{loss}}}{dz}$$

$$\gamma = \frac{1}{2 \left[\sqrt{\frac{\mu}{\epsilon}} \pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right) \right]} \left[\frac{\pi}{\sigma \delta} |H_0|^2 a \left[1 + \frac{a}{b} \right] \right]$$

$$\gamma = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} \frac{1}{\mu \sigma \delta} \frac{\left(\frac{1}{a} + \frac{1}{b} \right)}{\ln\left(\frac{b}{a}\right)}$$

(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z .

$$Z_0 = \frac{|\Delta \Phi|}{I_a}$$

$$Z_0 = \frac{\int_a^b \mathbf{E} \cdot \hat{\rho} d\rho}{I_a}$$

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}} H_0 a \frac{\int_a^b \frac{1}{\rho} d\rho}{I_a}$$

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}} H_0 a \frac{1}{I_a} \ln\left(\frac{b}{a}\right)$$

For a good conductor, the current flowing within the skin depth in from the surface can be treated as an effective current density on the surface because the skin depth is so small:

$$\mathbf{K}_{\text{eff}} = \mathbf{n} \times \mathbf{H}_{\parallel}$$

$$\mathbf{K}_{\text{eff}} = \hat{\boldsymbol{\rho}} \times \left[H_0 \frac{a}{\rho} \hat{\boldsymbol{\phi}} \right]_{\rho=a}$$

$$\mathbf{K}_{\text{eff}} = H_0 \hat{\mathbf{z}}$$

We find the total current by integrating the current density over the circumference of the cylinder:

$$I_a = 2\pi a H_0$$

Inserting this into the equation above, we find:

$$Z_0 = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln\left(\frac{b}{a}\right)$$

(d) The series resistance and inductance per unit length of the line can now be computed. For the total power dP lost after the wave has traversed a distance of the waveguide dz , that little stretch of waveguide can be thought of as a circuit element with resistance. Joule heating in an Ohmic circuit element obeys the law:

$$P_{\text{loss}} = I_{\text{rms}}^2 R$$

We have to be careful because the current I_{rms} in this equation is the root-mean-square current, and we have been dealing with peak values up until now. But for a harmonic wave, they are trivially related. The rms current is just $1/\sqrt{2}$ times the peak power for a harmonic wave:

$$P_{\text{loss}} = \frac{1}{2} I^2 R$$

For our incremental circuit element this becomes:

$$dP_{\text{loss}} = \frac{1}{2} I^2 dR$$

Divide both sides by the incremental length:

$$\frac{dP_{\text{loss}}}{dz} = \frac{1}{2} I^2 \frac{dR}{dz}$$

In accordance with the problem description, redefine R not as the total resistance, but as the resistance per unit length: $R \rightarrow \frac{dR}{dz}$:

$$\frac{dP_{\text{loss}}}{dz} = \frac{1}{2} I^2 R$$

$$R = \frac{2}{I^2} \frac{dP_{\text{loss}}}{dz}$$

$$R = \frac{2}{(2\pi a H_0)^2} \left[\frac{\pi}{\sigma \delta} |H_0|^2 a \left[1 + \frac{a}{b} \right] \right]$$

$$R = \frac{1}{2\pi \sigma \delta} \left[\frac{1}{a} + \frac{1}{b} \right]$$

The self-inductance can be thought of as a device's ability to store energy in its magnetic fields and is defined as:

$$W = \frac{1}{2} L I_{\text{rms}}^2 \quad \text{where } W \text{ is the total energy in the magnetic fields}$$

As done above, we convert the rms current to the peak current:

$$W = \frac{1}{4} L I^2$$

$$L = \frac{4}{I^2} W$$

We are only seeking the inductance per unit length, so we only need to find the energy in one two-dimensional slice. There are three separate regions, the inner conductor, the outer conductor, and the space between them. Each of them contributes to storing energy in the magnetic field.

$$L = \frac{4}{I^2} [W_{\text{in}} + W_{\text{mid}} + W_{\text{out}}]$$

$$L = \frac{4}{I^2} \left[\frac{1}{4\mu_c} \int |\mathbf{B}_{\text{in}}|^2 da + \frac{1}{4\mu} \int |\mathbf{B}_{\text{mid}}|^2 da + \frac{1}{4\mu_c} \int |\mathbf{B}_{\text{out}}|^2 da \right]$$

$$L = \frac{4}{I^2} \left[\frac{1}{4\mu_c} \int_0^{2\pi} \int_0^a |\mathbf{B}_{\text{in}}|^2 \rho d\rho d\phi + \frac{1}{4\mu} \int_0^{2\pi} \int_a^b |\mathbf{B}_{\text{mid}}|^2 \rho d\rho d\phi + \frac{1}{4\mu_c} \int_0^{2\pi} \int_b^\infty |\mathbf{B}_{\text{out}}|^2 \rho d\rho d\phi \right]$$

The inner and outer fields just die off exponentially into the conductors (radially) according to the skin depth. Be careful to remember that the field at the surface of the outer conductor is: $H(\rho=b) = \frac{a}{b} H_0$

$$L = \frac{2\pi}{I^2} |H_0|^2 \left[\mu_c \int_0^a e^{-2(a-\rho)/\delta} \rho d\rho + \mu a^2 \int_a^b \frac{1}{\rho} d\rho + \mu_c \frac{a^2}{b^2} \int_b^\infty e^{-2(\rho-b)/\delta} \rho d\rho \right]$$

$$L = \frac{2\pi}{I^2} |H_0|^2 \left[\mu_c e^{-2a/\delta} \int_0^a e^{2\rho/\delta} \rho d\rho + \mu a^2 \ln\left(\frac{b}{a}\right) + \mu_c \frac{a^2}{b^2} e^{2b/\delta} \int_b^\infty e^{-2\rho/\delta} \rho d\rho \right]$$

$$L = \frac{2\pi}{I^2} |H_0|^2 \left[\mu a^2 \ln\left(\frac{b}{a}\right) + \mu_c a^2 \left(\frac{1}{a} + \frac{1}{b}\right) \delta / 2 + \mu_c e^{-2a/\delta} \delta^2 / 4 \right]$$

$$L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) + \frac{\mu_c \delta}{4\pi} \left[\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{a^2} e^{-2a/\delta} \delta / 2 \right]$$

The skin depth is small compared to the radius of the inner conductor, so the last term goes away, leaving:

$$L = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right) + \frac{\mu_c \delta}{4\pi} \left(\frac{1}{a} + \frac{1}{b}\right)$$