Jackson 8.1 Homework Problem Solution

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PROBLEM:

Consider the electric and magnetic fields in the surface region of an excellent conductor in the approximation of Section 8.1, where the skin depth is very small compared to the radii of curvature of the surface or the scale of significant spatial variation of the fields just outside.

(a) For a single-frequency component, show that the magnetic field **H** and the current density **J** are such that **f**, the time-averaged force per unit area at the surface from the conduction current, is given by

$$\mathbf{f} = -\mathbf{n}\frac{\mu_c}{4}|\mathbf{H}_{\text{par}}|^2$$

where $\mathbf{H}_{\text{par}}$ is the peak parallel component of the magnetic field at the surface, μ_c is the magnetic permeability of the conductor, and **n** is the outward normal at the surface.

(b) If the magnetic permeability μ outside the surface is different from μ_c, is there an additional force per unit area? What about electric forces?

(c) Assume that the fields are a superposition of different frequencies (all high enough that the approximations still hold). Show that the time-averaged force takes the same form as in part with $|\mathbf{H}_{\text{par}}|^2$ replaced by $2\langle|\mathbf{H}_{\text{par}}|^2\rangle$ where the angle brackets <...> mean time average.

SOLUTION:

(a) The idea is that the an electrodynamic field outside a conductor induces a current in the conductor, and the current then feels a force when interacting with the magnetic field of the electrodynamic disturbance. Force can be thought of as a change in momentum or change in mechanical energy. When a wave strikes a good conductor, the conductor absorbs some of the energy and feels a force as a result. As discussed previously, the force **F** felt on a current **J** in a magnetic field **B** is:

$$\mathbf{F} = \int \mathbf{J}\times\mathbf{B}\, d^3x$$

If we want to know the force per unit area on a surface **f**, we drop two of the dimensions in the integral

$$\mathbf{f} = \int \mathbf{J}\times\mathbf{B}\, dx$$

Let us get this in terms of the **H** field inside the conductor

$$\mathbf{f} = \mu_c \int \mathbf{J}\times\mathbf{H}\, dx$$

Use Ohm's Law $\mathbf{J} = \sigma\mathbf{E}$

$$\mathbf{f}=\mu_c\sigma\int \mathbf{E}\times\mathbf{H}\,dx$$

It becomes obvious that the integrand is the Poynting vector. The force arises from the fact that energy flows into the conductor. We care about the time-averaged force, so that

$$\mathbf{f}=\mu_c\sigma\int <\mathbf{E}\times\mathbf{H}>dx$$

Inside a good conductor, we found the fields to be

$$\mathbf{H}_c=\mathbf{H}_{x=0}e^{i(x/\delta-\omega t)}e^{-x/\delta} \quad \text{and} \quad \mathbf{E}_c=\sqrt{\frac{\mu_c\omega}{\sigma}}\mathbf{n}\times\mathbf{H}_{x=0}e^{i(x/\delta+7\pi/4-\omega t)}e^{-x/\delta}$$

Plugging these in, being careful to only keep only the real part:

$$\mathbf{f}=\mu_c\sigma\int <\left[\sqrt{\frac{\mu_c\omega}{\sigma}}\mathbf{n}\times\mathbf{H}_{x=0}\cos(x/\delta+7\pi/4-\omega t)e^{-x/\delta}\right]\times\left[\mathbf{H}_{x=0}\cos(x/\delta-\omega t)e^{-x/\delta}\right]>dx$$

Realizing that $(\mathbf{n}\times\mathbf{H}_{x=0})\times\mathbf{H}_{x=0}=-\mathbf{n}|\mathbf{H}_{x=0}|^2$

$$\mathbf{f}=-\mathbf{n}\mu_c\sigma\sqrt{\frac{\mu_c\omega}{\sigma}}|\mathbf{H}_{x=0}|^2\int e^{-2x/\delta}<\cos(x/\delta+7\pi/4-\omega t)\cos(x/\delta-\omega t)>dx$$

Use a trigonometry addition formula to move the constant out of the cosine

$$\mathbf{f}=-\mathbf{n}\mu_c\sqrt{\frac{\mu_c\omega\sigma}{2}}|\mathbf{H}_{x=0}|^2\int e^{-2x/\delta}<(\cos(x/\delta-\omega t)+\sin(x/\delta-\omega t))\cos(x/\delta-\omega t)>dx$$

The time average of $\cos^2(A)$ is something we have already found to be ½. The time average of $\sin(A)\cos(A)$ is the time average of $\sin(2A)/2$ which is obviously 0 because the sine spends just as much time positive as negative. Also recognizing the skin depth $\delta=\sqrt{2/(\mu_c\omega\sigma)}$, this becomes:

$$\mathbf{f}=-\mathbf{n}\mu_c\frac{1}{2\delta}|\mathbf{H}_{x=0}|^2\int e^{-2x/\delta}dx$$

$$\mathbf{f}=-\mathbf{n}\mu_c\frac{1}{4}|\mathbf{H}_{x=0}|^2$$

(b) If the magnetic permeability μ outside the surface is different from μ_c, is there an additional force per unit area? What about electric forces?

Yes, there is an additional force. Part of the wave is reflected. A wave carries momentum and when it is reflected it changes momentum. A change in momentum is equivalent to a force being exerted. So the radiation outside the conductor and the conductor cause a force on each other due to this reflection. This is known as radiation pressure. The way to calculate radiation pressures is using the Maxwell stress tensor. Let us assume a transverse plane wave at normal incidence, for which the Maxwell stress tensor reduces to:

$$\mathbf{f} = \hat{\mathbf{n}} S_{xx} = \hat{\mathbf{n}} \left[\epsilon_0 |E_x|^2 + \frac{1}{\mu} |B_x|^2 - \frac{1}{2} \left(\epsilon_0 |\mathbf{E}|^2 + \frac{1}{\mu} |\mathbf{B}|^2 \right) \right]$$

$$\mathbf{f} = \hat{\mathbf{n}} \left[\epsilon_0 |E_x|^2 + \frac{1}{\mu} |B_x|^2 - \frac{1}{2} \left(\epsilon_0 |E_x|^2 + \epsilon_0 |E_t|^2 + \frac{1}{\mu} |B_x|^2 + \frac{1}{\mu} |B_t|^2 \right) \right]$$

The direction x is normal and into the conductor. For a transverse plane wave, there is no component of the fields in the direction of propagation, so that $E_x = 0$ and $B_x = 0$. For the approximations of this section, just outside the conductor the electric field can only be normal to the conductor's surface. This means that the transverse electric field must also be zero $E_t = 0$.

$$\mathbf{f} = -\mathbf{n} \frac{1}{2\mu} |B_t|^2$$

Let us get this in terms of **H** and note that the transverse component is what we meant by the component parallel to the conductor's surface.

$$\mathbf{f} = -\mathbf{n} \frac{1}{2} \mu |\mathbf{H}_{\text{par}}|^2$$

If we time average this we get the time-averaged force in terms of one half of the peak value.

$$\boxed{\mathbf{f} = -\mathbf{n} \frac{1}{4} \mu |\mathbf{H}_{x=0}|^2}$$

Note that there is no electric force due to the radiation pressure.

Let us find the electric force due to the induced sources:

$$\mathbf{F} = \int \rho \mathbf{E} \, d^3 x$$

For fields varying harmonically in time, the continuity equation becomes: $\rho = \frac{i}{\omega} \nabla \cdot \mathbf{J}$
Under the approximations of this section, the current points entirely in a direction parallel to the conductor's surface and varies spatially entirely in the direction perpendicular to the surface. This means that it has zero divergence. As a result there is no induced charge and no electric force.

(c) Assume that the fields are a superposition of different frequencies (all high enough that the approximations still hold). Show that the time-averaged force takes the same form as in part with $|\mathbf{H}_{\text{par}}|^2$ replaced by $2\langle|\mathbf{H}_{\text{par}}|^2\rangle$ where the angle brackets <...> mean time average.

With many frequencies involved, the time averaging step does not necessarily equate to one half times the peak value as it did for the single-frequency case. Without knowing the detailed frequencies involved, the best we can do is just leave the time-average unevaluated. In a sense, you just start with our end answer to part a and you reverse the time-averaging step to get the general form. Because the time-averaging of the single frequency case evaluated to half of the peak value, we just multiply by 2 and by the angle brackets back to get back to the general case.