



C. S. BAIRD

## Jackson 7.6 Homework Problem Solution

Dr. Christopher S. Baird  
University of Massachusetts Lowell



### **PROBLEM:**

A plane wave of frequency  $\omega$  is incident normally from vacuum on a semi-infinite slab of material with a *complex* index of refraction  $n(\omega)$  [ $n^2(\omega) = \epsilon(\omega)/\epsilon_0$ ].

a) Show that the ratio of reflected power to incident power is

$$R = \left| \frac{1 - n(\omega)}{1 + n(\omega)} \right|^2$$

while the ratio of power transmitted into the medium to the incident power is

$$T = \frac{4\Re(n(\omega))}{|1 + n(\omega)|^2}$$

b) Evaluate  $\Re[i\omega(\mathbf{E} \cdot \mathbf{D}^* - \mathbf{B} \cdot \mathbf{H}^*)/2]$  as a function of  $(x, y, z)$ . Show that this rate of change of energy per unit volume accounts for the relative transmitted power  $T$ .

c) For a conductor, with  $n^2 = 1 + i(\sigma/\omega\epsilon_0)$ ,  $\sigma$  real, write out the results of parts a and b in the limit  $\epsilon_0\omega \ll \sigma$ . Express your answer in terms of  $\delta$  as much as possible. Calculate  $\frac{1}{2}\Re(\mathbf{J}^* \cdot \mathbf{E})$  and compare with the results of part b. Do both enter the complex form of Poynting's theorem?

### **SOLUTION:**

(a) The corrected Fresnel equations were presented in the lecture notes, and they are all we need to solve this problem. However, let us go into the details of their derivation to make clear where different parts come from.

A plane, linearly polarized, monochromatic electromagnetic wave traveling in the  $\mathbf{k}$  direction in a linear, lossy medium has wave number  $k = \beta + i\alpha/2$  where  $k = \omega\sqrt{\mu\epsilon}$ , and has an electric field:

$$\mathbf{E} = E_0 \boldsymbol{\epsilon}_1 e^{-\alpha \hat{\mathbf{k}} \cdot \mathbf{x}/2} e^{i\beta \hat{\mathbf{k}} \cdot \mathbf{x} - i\omega t + i\theta}$$

Here,  $E_0$  is the real-valued magnitude of the electric field at  $z = 0$  and time  $t = \theta/\omega$ ,  $\boldsymbol{\epsilon}_1$  is the real-valued polarization unit vector,  $\alpha$  is twice the imaginary part of the wave number or attenuation constant,  $\beta$  is the real part of the wave number,  $\theta$  is the phase of the wave, and  $\omega$  is the temporal angular frequency.

Inserting this into Faraday's law leads directly to the associated magnetic field

$$\mathbf{B} = \frac{n}{c} E_0 (\hat{\mathbf{k}} \times \boldsymbol{\epsilon}_1) e^{-\alpha \hat{\mathbf{k}} \cdot \mathbf{x}/2} e^{i\beta \hat{\mathbf{k}} \cdot \mathbf{x} - i\omega t + i\theta} \quad \text{where the index of refraction is complex: } n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$$

The time-averaged complex-valued power density transmitted by the wave in the  $\mathbf{k}$  direction is the complex Poynting vector:

$$\hat{\mathbf{k}} \cdot \mathbf{S} = \frac{1}{2\mu} \hat{\mathbf{k}} \cdot \mathbf{E} \times \mathbf{B}^*$$

Insert the electric and magnetic field expressions above to find:

$$\hat{\mathbf{k}} \cdot \mathbf{S} = \frac{n^*}{2\mu c} e^{-\alpha \hat{\mathbf{k}} \cdot \mathbf{x}} |E_0|^2$$

The fraction of incident power  $T$  transmitted through a surface is the total power of the wave at the surface on the transmitted side divided by the total incident power at the surface on the incident side:

$$T = \frac{\Re(\hat{\mathbf{k}} \cdot \mathbf{S}_t(\mathbf{x}=0)) A_t}{\Re(\hat{\mathbf{k}} \cdot \mathbf{S}_i(\mathbf{x}=0)) A_i}$$

Even though we are dealing with plane waves, for the purpose of calculating the areas, pretend we have square beams. The depth of the incident and transmitted beam will be the same, but the widths will be different because refraction causes the transmitted beam to spread out. In order to calculate the ratio of widths, we must relate the two widths. Let us relate them both to the projected width  $w$ , which is the same for both, as shown in the figure:  $w_i = w \cos \theta_i$  and  $w_t = w \cos \theta_t$ . Then:

$$\frac{A_t}{A_i} = \frac{l w_t}{l w_i} = \frac{l w \cos \theta_t}{l w \cos \theta_i} = \frac{\cos \theta_t}{\cos \theta_i}$$

Use Snell's law to get rid of the transmitted angle:

$$\frac{A_t}{A_i} = \frac{\sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{\cos \theta_i}$$

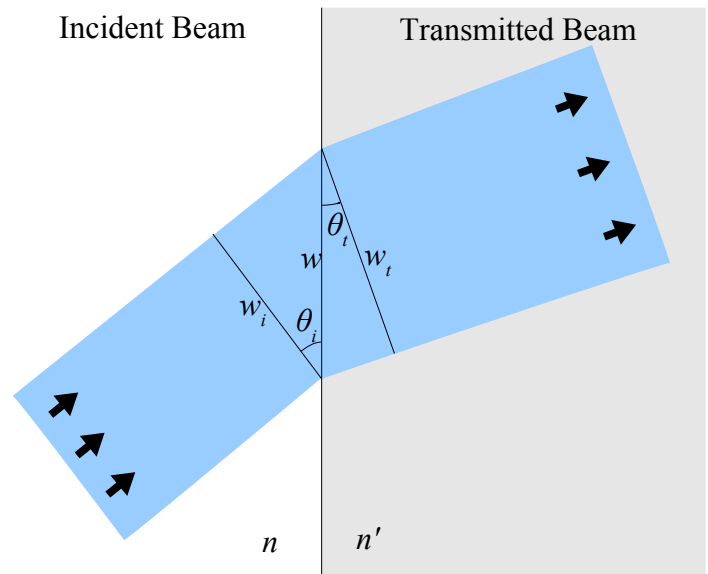
Using this ratio, the transmission becomes:

$$T = \frac{\Re(\hat{\mathbf{k}} \cdot \mathbf{S}_t(\mathbf{x}=0)) \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{\Re(\hat{\mathbf{k}} \cdot \mathbf{S}_i(\mathbf{x}=0)) \cos \theta_i}$$

Insert the power densities to find:

$$T = \frac{\Re(n') \mu \sqrt{1 - (n/n')^2 \sin^2 \theta_i} |E_0'|^2}{\Re(n) \mu' \cos \theta_i |E_0|^2}$$

Similarly, we find the fraction of power reflected:



$$R = \frac{P_r(\mathbf{x}=0)}{P_i(\mathbf{x}=0)}$$

$$R = \left| \frac{E_0^r}{E_0} \right|^2$$

All of the factors out front cancel in this case because the incident and reflected beams are in the same material. At this point we can use the electric field strength ratios derived in class to find the general Fresnel equations:

For electric fields always perpendicular to the plane of incidence:

$$R = \left| \frac{n \cos \theta_i - n' \frac{\mu}{\mu'} \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{n^* \cos \theta_i + n' \frac{\mu}{\mu'} \sqrt{1 - (n/n')^2 \sin^2 \theta_i}} \right|^2$$

*Polarization Perpendicular  
to the Plane of Incidence*

$$T = \left| \frac{\Re(n') \frac{\mu}{\mu'} \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{\Re(n) \mu' \cos \theta_i} \frac{2 \Re(n) \cos \theta_i}{n^* \cos \theta_i + n' \frac{\mu}{\mu'} \sqrt{1 - (n/n')^2 \sin^2 \theta_i}} \right|^2$$

For electric fields always parallel to the plane of incidence:

$$R = \left| \frac{n' \cos \theta_i - n \frac{\mu'}{\mu} \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{n' \cos \theta_i + n \frac{\mu'}{\mu} \sqrt{1 - (n/n')^2 \sin^2 \theta_i}} \right|^2$$

*Polarization Parallel to the  
Plane of Incidence*

$$T = \left| \frac{\Re(n') \frac{\mu}{\mu'} \sqrt{1 - (n/n')^2 \sin^2 \theta_i}}{\Re(n) \mu' \cos \theta_i} \frac{2 \Re(n) \cos \theta_i}{n' \frac{\mu}{\mu'} \cos \theta_i + n^* \sqrt{1 - (n/n')^2 \sin^2 \theta_i}} \right|^2$$

These are the most general Fresnel equations.

In this problem the material is evidently non-magnetic,  $\mu' = \mu$ , and we are at normal incidence,  $\theta_i = 0$ , and the incident material is free space so that  $n = 1$ ,  $n' = n(\omega)$ . Plugging these into the Fresnel equations reveals:

Perpendicular to plane of incidence:

$$R = \left| \frac{1-n(\omega)}{1+n(\omega)} \right|^2 \quad T = \frac{4\Re(n(\omega))}{|1+n(\omega)|^2}$$

Parallel to plane of incident:

$$R = \left| \frac{1-n(\omega)}{1+n(\omega)} \right|^2 \quad T = \frac{4\Re(n(\omega))}{|1+n(\omega)|^2}$$

The fact that both polarizations match at normal incidence is a reassuring consistency check.

b) We now examine rate of change of energy density:  $\Re [i\omega(\mathbf{E} \cdot \mathbf{D}^* - \mathbf{B} \cdot \mathbf{H}^*)/2]$ .

The material is linear and we are dealing with single-frequency plane waves.

We use:

$$\mathbf{D} = \epsilon(\omega)\mathbf{E}, \quad \omega\mathbf{B} = \mathbf{k} \times \mathbf{E}, \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$$

to find:

$$\Re [i\omega(\mathbf{E} \cdot \mathbf{D}^* - \mathbf{B} \cdot \mathbf{H}^*)/2] = \Re \left[ \frac{i\omega}{2} (\epsilon(\omega)^* |\mathbf{E}|^2 - \frac{1}{\mu_0 \omega^2} |\mathbf{k} \times \mathbf{E}|^2) \right]$$

The incident wave in this problem is a transverse plane wave propagating normal to the material's surface, so the transmitted wave must also be a transverse plane wave propagating normal due to symmetry. Define the  $z$  dimension as normal and into the material, and the  $x$  axis as the polarization direction. The electric field inside the material takes on the form

$$\mathbf{E} = E_t \hat{\mathbf{x}} e^{ikz - i\omega t}$$

We must be careful to realize with a complex-valued index of refraction, the wave number is also complex. This means the imaginary part of the wave number will lead to a damped exponential factor which is part of the expression's magnitude, not its phase. We therefore split up  $k$  into real and imaginary parts:

$$\mathbf{E} = E_t \hat{\mathbf{x}} e^{-\Im(k)z} e^{i\Re(k)z - i\omega t}$$

This leads to:

$$|\mathbf{E}|^2 = |E_t|^2 e^{-2\Im(k)z} \quad \text{and} \quad |\mathbf{k} \times \mathbf{E}|^2 = |k|^2 |E_t|^2 e^{-2\Im(k)z}$$

Using all of these in the energy density expression, we find:

$$\begin{aligned}
& \Re [i \omega (\mathbf{E} \cdot \mathbf{D}^* - \mathbf{B} \cdot \mathbf{H}^*) / 2] \\
&= \Re \left[ \frac{i \omega}{2} |E_i|^2 e^{-2 \Im(k) z} \left( \epsilon(\omega)^* - \frac{1}{\mu_0 \omega^2} |k|^2 \right) \right] \\
&= \Re \left[ \frac{i \omega \epsilon_0}{2} |E_i|^2 e^{-2 \omega \sqrt{\mu_0 \epsilon_0} \Im(n(\omega)) z} \left( (n^2(\omega))^* - |n(\omega)|^2 \right) \right] \\
&= \boxed{\omega \epsilon_0 |E_i|^2 e^{-2 \omega \Im(n(\omega)) z / c} \Re(n) \Im(n)}
\end{aligned}$$

The power per unit area transmitted into the material is just the integral of the change in energy density:

$$\begin{aligned}
\frac{P_t}{A} &= \int_0^{\infty} \Re [i \omega (\mathbf{E} \cdot \mathbf{D}^* - \mathbf{B} \cdot \mathbf{H}^*) / 2] dz \\
\frac{P_t}{A} &= \int_0^{\infty} \omega \epsilon_0 |E_i|^2 e^{-2 \omega \Im(n(\omega)) z / c} \Re(n) \Im(n) dz \\
\frac{P_t}{A} &= \omega \epsilon_0 |E_i|^2 \Re(n) \Im(n) \int_0^{\infty} e^{-2 \omega \Im(n(\omega)) z / c} dz \\
\frac{P_t}{A} &= \epsilon_0 |E_i|^2 \Re(n) \frac{c}{2}
\end{aligned}$$

The transmission coefficient  $T$  is the relative power transmitted:

$$\begin{aligned}
T &= \frac{P_t / A}{P_i / A} = \frac{\epsilon_0 |E_i|^2 \Re(n) \frac{c}{2}}{\frac{1}{2} \sqrt{\epsilon_0 / \mu_0} |E_i|^2} \\
T &= \Re(n) \frac{|E_t|^2}{|E_i|^2}
\end{aligned}$$

We plug in the ratio of the field intensities as found in the lecture notes:

$$\boxed{T = \frac{4 \Re(n(\omega))}{|1 + n(\omega)|^2}}$$

This matches the answer found in part a.

c) For a good conductor we model the index of refraction as:  $n = \sqrt{1 + i \sigma / \omega \epsilon_0}$

To see how this applies to the reflection and transmission coefficient, we need to break the index of refraction into real and imaginary parts. We use the identity:

$$\sqrt{z} = (\sqrt{|z| + \Re(z)} + \operatorname{sgn}(\Im(z))i\sqrt{|z| - \Re(z)})/\sqrt{2}$$

to find:

$$n = (\sqrt{\sqrt{1 + \sigma^2/\omega^2 \epsilon_0^2} + 1} + i\sqrt{\sqrt{1 + \sigma^2/\omega^2 \epsilon_0^2} - 1})/\sqrt{2}$$

For a good conductor, we have  $\epsilon_0\omega \ll \sigma$  which is the same as  $\sigma/\epsilon_0\omega \gg 1$  and  $\frac{c}{\omega\delta} \gg 1$  where

$$\delta = \sqrt{\frac{2}{\mu_c \omega \sigma}}$$

We can use these to simplify the dielectric constant, and thereby the reflection and transmission coefficients. When there is a sum of  $\sigma/\epsilon_0\omega$  and 1, we can simply drop the one.

$$n \approx \sqrt{\sigma/2\omega\epsilon_0}(1+i)$$

In terms of the skin depth, this becomes:

$$n \approx \frac{c}{\delta\omega}(1+i) \quad \text{where}$$

Now plug this index of refraction into the transmission and reflection coefficients:

$$R = \left| \frac{1 - n(\omega)}{1 + n(\omega)} \right|^2$$

$$R \approx \left| \frac{1 - \frac{c}{\delta\omega}(1+i)}{1 + \frac{c}{\delta\omega}(1+i)} \right|^2$$

Evaluate the magnitude squared:

$$R \approx \frac{1 - 2\frac{c}{\delta\omega} + 2\frac{c^2}{\delta^2\omega^2}}{1 + 2\frac{c}{\delta\omega} + 2\frac{c^2}{\delta^2\omega^2}}$$

Use  $\frac{c}{\omega\delta} \gg 1$  to simplify this to:

$$R \approx 1 - 2\frac{\delta\omega}{c}$$

$$T \approx \frac{4\Re(n(\omega))}{|1 + n(\omega)|^2}$$

$$T \approx \frac{2\delta\omega}{c}$$

The change in energy density is:

$$\frac{1}{\mu_0 \delta^2 \omega} |E_i|^2 e^{-2\frac{z}{\delta}}$$

which reveals why the skin depth of a conductor is defined as it is.

Calculating the energy density lost to Ohmic heating:

$$\begin{aligned} & \frac{1}{2} \Re(\mathbf{J}^* \cdot \mathbf{E}) \\ &= \frac{\sigma}{2} \Re(\mathbf{E}^* \cdot \mathbf{E}) \\ &= \frac{\sigma}{2} |E_i|^2 e^{-2\omega \Im(n)z/c} \\ &= \frac{1}{\mu_0 \delta^2 \omega} |E_i|^2 e^{-2\frac{z}{\delta}} \end{aligned}$$

This matches exactly the change in energy density, telling us that in the limit if a thick good conductor, all of the energy that is not reflected is lost to ohmic heating. Both of these expressions do not enter into Poynting's theorem because the both express the same energy loss mechanism.