PROBLEM:
A plane-polarized electromagnetic wave of frequency $\omega$ in free space is incident normally on the flat surface of a non-permeable medium of conductivity $\sigma$ and dielectric constant $\varepsilon$.

(a) Calculate the amplitude and phase of the reflected wave relative to the incident wave for arbitrary $\sigma$ and $\varepsilon$.

(b) Discuss the limiting case of a very poor and a very good conductor, and show that for a good conductor the reflection coefficient (ratio of reflected to incident intensity) is approximately

$$R \approx 1 - 2 \frac{\omega}{c} \delta$$

where $\delta$ is the skin depth.

SOLUTION:
(a) If we set up the problem in the usual way with the incident wave $E$ traveling in the positive $z$ direction, the transmitted wave $E'$ traveling in the positive $z$ direction, and the reflected wave $E''$ traveling in the negative $z$ direction, we get from the boundary conditions:

$$E_0' = E_0 + E_0''$$

$$k' E_0' = k E_0 - k E_0''$$

We can use both of these to solve for the reflected wave in terms of the incident wave:

$$E_0'' = \frac{k}{k'} E_0 - \frac{k}{k'} E_0'' - E_0$$

$$E_0'' = \frac{k - k'}{k + k'} E_0$$

The wave number in free space is just $k = \frac{\omega}{c}$

The wave number in the other material obeys: $k' = \sqrt{\varepsilon(\omega) \mu_0} \omega$

For a conducting material we can use $\varepsilon(\omega) = \varepsilon + i \frac{\sigma}{\omega}$ so that we have
\[ k' = \sqrt{\varepsilon \mu_0 + i \frac{\sigma}{\varepsilon_0 \omega} \mu_0 \omega} \]

Plugging these in above leads to:

\[ \frac{E_0''}{E_0} = \frac{1 - \sqrt{\frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega}}}{1 + \sqrt{\frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega}}} \]

The problem asks us to explicitly expand this in terms of amplitude and phase. The amplitude of a complex number is defined as \(|z| = \sqrt{z^*z}\) so that we have

\[
\left| \frac{E_0''}{E_0} \right| = \sqrt{\left| 1 - \sqrt{\frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega}} \right|^2 - \left| \frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega} \right|^2}
\]

\[
\left| \frac{E_0''}{E_0} \right| = \sqrt{\left| 1 + \sqrt{\frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega}} \right|^2 - \left| \frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega} \right|^2}
\]

The phase of a complex number is \(\text{Arg}(z) = \tan^{-1}\left( -\frac{\text{Im}(z)}{\text{Re}(z)} \right)\)

\[
\text{Arg}\left( \frac{E_0''}{E_0} \right) = \tan^{-1}\left( \frac{2 \text{Im} \sqrt{\frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega}}}{1 - \sqrt{\frac{\varepsilon}{\varepsilon_0} + i \frac{\sigma}{\varepsilon_0 \omega}}^2} \right)
\]

(b) Discuss the limiting case of a very poor and a very good conductor, and show that for a good conductor the reflection coefficient (ratio of reflected to incident intensity) is approximately

\[ R \approx 1 - 2 \frac{\omega}{c} \delta \]

where \(\delta\) is the skin depth.

For a very poor conductor \(\sigma \ll \varepsilon_0 \omega\) which means the same as \(\sigma/\varepsilon_0 \omega \ll 1\). This leads to:
\[ \frac{|E_0''|}{E_0} = \sqrt{\frac{1 + \frac{\epsilon}{\epsilon_0} - 2 \sqrt{\frac{\epsilon}{\epsilon_0}}}{1 + \frac{\epsilon}{\epsilon_0} + 2 \sqrt{\frac{\epsilon}{\epsilon_0}}} \frac{\epsilon}{\epsilon_0}} \]

\[ \frac{|E_0''|}{E_0} = \frac{1 - \frac{\epsilon}{\epsilon_0}}{1 + \frac{\epsilon}{\epsilon_0}} \]

Now recognize:

\[ \Im \sqrt{x + iy} = \Im (x^2 + y^2)^{1/4} e^{\frac{i}{2} \tan^{-1}(y/x)} \]

\[ \Im \sqrt{x + iy} = (x^2 + y^2)^{1/4} \sin \left( \frac{1}{2} \tan^{-1}(y/x) \right) \]

If \( y \ll x \) then \( \tan^{-1}(y/x) = y/x \) and \( \sin \left( \frac{1}{2} \frac{y}{x} \right) = \frac{1}{2} \frac{y}{x} \) so that

\[ \Im \sqrt{x + iy} = \frac{1}{2} \sqrt{x/y} \]

leading to:

\[ \text{Arg} \left( \frac{E_0''}{E_0} \right) = \tan^{-1} \left( \frac{\sqrt{\frac{\epsilon}{\epsilon_0}} \frac{\sigma}{\omega}}{\sqrt{\frac{\epsilon}{\epsilon_0} - 1}} \right) \]

We must be careful because the arctan leads to some ambiguity of what quadrant the phase should be in. We know from physical arguments that the phase difference should be \( \pi \) for a perfect dielectric, so that our answer should reduce to \( \pi \) for \( \sigma \rightarrow 0 \). The answer must be in the third quadrant:

\[ \text{Arg} \left( \frac{E_0''}{E_0} \right) = \pi + \tan^{-1} \left( \frac{\sqrt{\frac{\epsilon}{\epsilon_0}} \frac{\sigma}{\omega}}{\sqrt{\frac{\epsilon}{\epsilon_0} - 1}} \right) \]

For a very good conductor \( \sigma \gg \epsilon_0 \omega \)

\[ \frac{|E_0''|}{E_0} = \sqrt{\frac{1 + \frac{\sigma}{\epsilon_0 \omega} - \sqrt{2} \sqrt{\frac{\sigma}{\epsilon_0 \omega}}}{1 + \frac{\sigma}{\epsilon_0 \omega} + \sqrt{2} \sqrt{\frac{\sigma}{\epsilon_0 \omega}}}} \]

which leads to

\[ \frac{|E_0''|}{E_0} = \sqrt{\frac{1 - \sqrt{2} \sqrt{\frac{\epsilon_0 \omega}{\sigma}}}{1 + \sqrt{2} \sqrt{\frac{\epsilon_0 \omega}{\sigma}}}} \]

after dropping small terms
Use the binomial expansion $\sqrt{1+x} = 1 + (1/2)x + ...$ on top and bottom and drop all small terms

$$\frac{|E_0''|}{E_0} = 1 - \frac{1}{2} \frac{\varepsilon_0 \omega}{\sigma}$$

which leads to

$$\frac{|E_0''|}{E_0} = 1 + \frac{1}{2} \frac{\varepsilon_0 \omega}{\sigma} - \sqrt{2} \frac{\varepsilon_0 \omega}{\sigma}$$

We can now calculate the reflection coefficient:

$$R = \frac{|E_0''|^2}{|E_0|^2} = \left(1 - \sqrt{2} \frac{\varepsilon_0 \omega}{\sigma}\right)^2 = 1 + 2 \frac{\varepsilon_0 \omega}{\sigma} - 2 \sqrt{2} \frac{\varepsilon_0 \omega}{\sigma}$$

Drop the smallest term

$$R = 1 - 2 \sqrt{2} \frac{\varepsilon_0 \omega}{\sigma}$$

or

$$R = 1 - 2 \frac{\omega}{c}$$

where $\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}$

The phase for a good conductor can also be found:

$$\arg \left(\frac{E_0''}{E_0}\right) = \tan^{-1} \left(\sqrt{2} \frac{\sigma}{\varepsilon_0 \omega} \frac{\varepsilon_0 \omega}{\sigma - 1}\right)$$

Drop the 1 in the denominator because it is much smaller:

$$\arg \left(\frac{E_0''}{E_0}\right) = \tan^{-1} \left(\sqrt{2} \frac{\varepsilon_0 \omega}{\sigma}\right)$$

$$\arg \left(\frac{E_0''}{E_0}\right) = \pi + \sqrt{\frac{2 \varepsilon_0 \omega}{\sigma}}$$