Jackson 7.4 Homework Problem Solution

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PROBLEM:

A plane-polarized electromagnetic wave of frequency ω in free space is incident normally on the flat surface of a non-permeable medium of conductivity σ and dielectric constant ε.

(a) Calculate the amplitude and phase of the reflected wave relative to the incident wave for arbitrary σ and ε.

(b) Discuss the limiting case of a very poor and a very good conductor, and show that for a good conductor the reflection coefficient (ratio of reflected to incident intensity) is approximately

$$R \approx 1 - 2\frac{\omega}{c}\delta$$

where δ is the skin depth.

SOLUTION:

(a) If we set up the problem in the usual way with the incident wave **E** traveling in the positive z direction, the transmitted wave **E'** traveling in the positive z direction, and the reflected wave **E''** traveling in the negative z direction, we get from the boundary conditions:

$$E_0' = E_0 + E_0''$$

$$k' E_0' = k E_0 - k E_0''$$

We can use both of these to solve for the reflected wave in terms of the incident wave:

$$E_0'' = \frac{k}{k'} E_0 - \frac{k}{k'} E_0'' - E_0$$

$$E_0'' = \frac{k - k'}{k + k'} E_0$$

The wave number in free space is just $k = \frac{\omega}{c}$

The wave number in the other material obeys: $k' = \sqrt{\epsilon(\omega)\mu_0}\,\omega$

For a conducting material we can use $\epsilon(\omega) = \epsilon + i\frac{\sigma}{\omega}$ so that we have

$$k' = \sqrt{\epsilon \mu_0 + i \frac{\sigma}{\omega} \mu_0 \omega}$$

Plugging these in above leads to:

$$\frac{E_0''}{E_0} = \frac{1 - \sqrt{\frac{\epsilon}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega}}}{1 + \sqrt{\frac{\epsilon}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega}}}$$

The problem asks us to explicitly expand this in terms of amplitude and phase. The amplitude of a complex number is defined as $|z| = \sqrt{z^* z}$ so that we have

$$\left| \frac{E_0''}{E_0} \right| = \sqrt{\left(\frac{1 - \sqrt{\frac{\epsilon}{\epsilon_0} - i \frac{\sigma}{\epsilon_0 \omega}}}{1 + \sqrt{\frac{\epsilon}{\epsilon_0} - i \frac{\sigma}{\epsilon_0 \omega}}} \right) \left(\frac{1 - \sqrt{\frac{\epsilon}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega}}}{1 + \sqrt{\frac{\epsilon}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega}}} \right)}$$

$$\boxed{\left| \frac{E_0''}{E_0} \right| = \sqrt{\frac{1 + \sqrt{\frac{\epsilon^2}{\epsilon_0^2} + \frac{\sigma^2}{\epsilon_0^2 \omega^2}} - 2 \Re \sqrt{\frac{\epsilon}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega}}}{1 + \sqrt{\frac{\epsilon^2}{\epsilon_0^2} + \frac{\sigma^2}{\epsilon_0^2 \omega^2}} + 2 \Re \sqrt{\frac{\epsilon}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega}}}}}$$

The phase of a complex number is $Arg(z) = \tan^{-1}\left(-i \frac{z - z^*}{z + z^*} \right)$

$$\boxed{Arg\left(\frac{E_0''}{E_0} \right) = \tan^{-1}\left(\frac{2 \Im \sqrt{\frac{\epsilon}{\epsilon_0} - i \frac{\sigma}{\epsilon_0 \omega}}}{1 - \sqrt{\frac{\epsilon^2}{\epsilon_0^2} + \frac{\sigma^2}{\epsilon_0^2 \omega^2}}} \right)}$$

(b) Discuss the limiting case of a very poor and a very good conductor, and show that for a good conductor the reflection coefficient (ratio of reflected to incident intensity) is approximately

$$R \approx 1 - 2 \frac{\omega}{c} \delta$$

where δ is the skin depth.

For a very poor conductor $\sigma \ll \varepsilon_0 \omega$ which means the same as $\sigma / \varepsilon_0 \omega \ll 1$. This leads to:

$$\left|\frac{E_0''}{E_0}\right| = \sqrt{\frac{1+\frac{\epsilon}{\epsilon_0}-2\sqrt{\frac{\epsilon}{\epsilon_0}}}{1+\frac{\epsilon}{\epsilon_0}+2\sqrt{\frac{\epsilon}{\epsilon_0}}}}$$

$$\boxed{\left|\frac{E_0''}{E_0}\right| = \frac{1-\sqrt{\frac{\epsilon}{\epsilon_0}}}{1+\sqrt{\frac{\epsilon}{\epsilon_0}}}}$$

Now recognize:

$$\Im\sqrt{x+i\,y} = \Im\,(x^2+y^2)^{1/4}\,e^{\frac{i}{2}\tan^{-1}(y/x)}$$

$$\Im\sqrt{x+i\,y} = (x^2+y^2)^{1/4}\sin\left(\frac{1}{2}\tan^{-1}(y/x)\right)$$

If $y << x$ then $\tan^{-1}(y/x) = y/x$ and $\sin(\frac{1}{2}\,y/x) = \frac{1}{2}\,y/x$ so that

$$\Im\sqrt{x+i\,y} = \frac{1}{2}\sqrt{x}\,y/x \quad \text{leading to:}$$

$$Arg\left(\frac{E_0''}{E_0}\right) = \tan^{-1}\left(\frac{\sqrt{\frac{\epsilon}{\epsilon_0}}\frac{\sigma}{\epsilon\,\omega}}{\frac{\epsilon}{\epsilon_0}-1}\right)$$

We must be careful because the arctan leads to some ambiguity of what quadrant the phase should be in. We know from physical arguments that the phase difference should be π for a perfect dielectric, so that our answer should reduce to π for $\sigma\to 0$. The answer must be in the third quadrant:

$$\boxed{Arg\left(\frac{E_0''}{E_0}\right) = \pi + \frac{\sqrt{\frac{\epsilon}{\epsilon_0}}}{\frac{\epsilon}{\epsilon_0}-1}\frac{\sigma}{\epsilon\,\omega}}$$

For a very good conductor $\sigma >> \varepsilon_0\omega$

$$\left|\frac{E_0''}{E_0}\right| = \sqrt{\frac{1+\frac{\sigma}{\epsilon_0\,\omega}-\sqrt{2}\sqrt{\frac{\sigma}{\epsilon_0\omega}}}{1+\frac{\sigma}{\epsilon_0\,\omega}+\sqrt{2}\sqrt{\frac{\sigma}{\epsilon_0\omega}}}} \quad \text{which leads to} \quad \left|\frac{E_0''}{E_0}\right| = \sqrt{\frac{1-\sqrt{2}\sqrt{\frac{\epsilon_0\,\omega}{\sigma}}}{1+\sqrt{2}\sqrt{\frac{\epsilon_0\,\omega}{\sigma}}}} \quad \text{after dropping small terms}$$

Use the binomial expansion $\sqrt{1+x}=1+(1/2)x+...$ on top and bottom and drop all small terms

$$\left|\frac{E_0''}{E_0}\right|=\frac{1-\frac{1}{2}\sqrt{2}\sqrt{\frac{\epsilon_0\omega}{\sigma}}}{1+\frac{1}{2}\sqrt{2}\sqrt{\frac{\epsilon_0\omega}{\sigma}}} \quad \text{which leads to} \quad \left|\frac{E_0''}{E_0}\right|=\frac{1+\frac{1}{2}\frac{\epsilon_0\omega}{\sigma}-\sqrt{2}\sqrt{\frac{\epsilon_0\omega}{\sigma}}}{1-\frac{1}{2}\frac{\epsilon_0\omega}{\sigma}}$$

$$\boxed{\left|\frac{E_0''}{E_0}\right|=1-\sqrt{2}\sqrt{\frac{\epsilon_0\omega}{\sigma}}}$$

We can now calculate the reflection coefficient:

$$R=\left|\frac{E_0''}{E_0}\right|^2=\left(1-\sqrt{2}\sqrt{\frac{\epsilon_0\omega}{\sigma}}\right)^2=1+2\frac{\epsilon_0\omega}{\sigma}-2\sqrt{2}\sqrt{\frac{\epsilon_0\omega}{\sigma}}$$

Drop the smallest term

$$\boxed{R=1-2\sqrt{2}\sqrt{\frac{\epsilon_0\omega}{\sigma}}} \quad \text{or} \quad \boxed{R=1-2\frac{\omega}{c}\delta} \quad \text{where} \quad \delta=\sqrt{\frac{2}{\mu_0\sigma\omega}}$$

The phase for a good conductor can also be found:

$$Arg\left(\frac{E_0''}{E_0}\right)=\tan^{-1}\left(\frac{\sqrt{2}\sqrt{\frac{\sigma}{\epsilon_0\omega}}}{\frac{\sigma}{\epsilon_0\omega}-1}\right)$$

Drop the 1 in the denominator because it is much smaller:

$$Arg\left(\frac{E_0''}{E_0}\right)=\tan^{-1}\left(\sqrt{2}\sqrt{\frac{\epsilon_0\omega}{\sigma}}\right)$$

$$\boxed{Arg\left(\frac{E_0''}{E_0}\right)=\pi+\sqrt{\frac{2\epsilon_0\omega}{\sigma}}}$$