PROBLEM:
Use the Kramers-Kronig relation to calculate the real part of $\varepsilon(\omega)$, given the imaginary part of $\varepsilon(\omega)$ for positive ω as

(a) $\Im(\epsilon/\epsilon_0)=\lambda[\theta(\omega-\omega_1)-\theta(\omega-\omega_2)]$, $\omega_2 > \omega_1 > 0$

(b) $\Im(\epsilon/\epsilon_0)=\dfrac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

SOLUTION:
(a) Inserting the first expression for the imaginary part:

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2}{\pi}P\int_0^\infty \frac{\Im(\epsilon(\omega')/\epsilon_0)\omega'}{\omega'^2-\omega^2}d\omega'$$

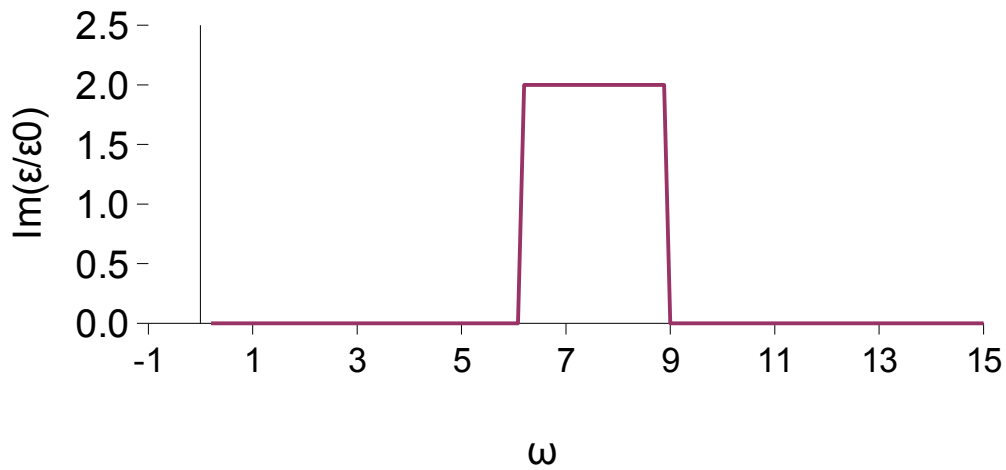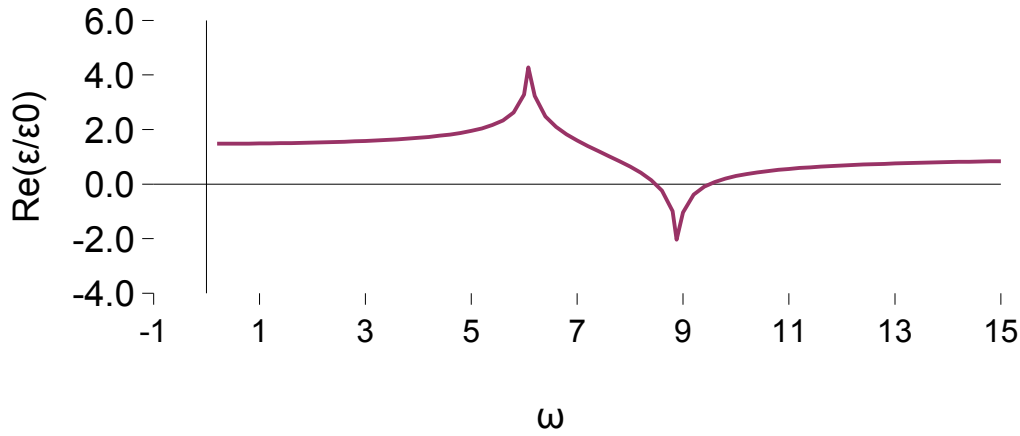$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2}{\pi}P\int_0^\infty \frac{\lambda[\theta(\omega'-\omega_1)-\theta(\omega'-\omega_2)]\omega'}{\omega'^2-\omega^2}d\omega'$$

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2\lambda}{\pi}P\int_{\omega_1}^\infty \frac{\omega'}{\omega'^2-\omega^2}d\omega'-\frac{2\lambda}{\pi}P\int_{\omega_2}^\infty \frac{\omega'}{\omega'^2-\omega^2}d\omega'$$

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2\lambda}{\pi}\left[\frac{1}{2}\ln\left(|\omega'^2-\omega^2|\right)\right]_{\omega_1}^\infty-\frac{2\lambda}{\pi}\left[\frac{1}{2}\ln\left(|\omega'^2-\omega^2|\right)\right]_{\omega_2}^\infty$$

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{\lambda}{\pi}\left[\ln\left(|\omega_2^2-\omega^2|\right)-\ln\left(|\omega_1^2-\omega^2|\right)\right]$$

We can plot these to see how they compare to the harmonic model (as shown on the next page)

This looks quite similar to the results from the harmonic model. The difference is that the flat top of the imaginary part of the dielectric constant leads to an extended region of anomalous dispersion in the real part. Also the unrealistically steep walls of the imaginary part leads to infinite peaks in the real part.

(b) Using $\Im(\epsilon/\epsilon_0)=\dfrac{\lambda\,\gamma\,\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$ we have:

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2\lambda\gamma}{\pi}P\int_0^\infty \frac{\omega'^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)(\omega'^2-\omega^2)}\,d\omega'$$

Let us use partial fraction decomposition to break up the fraction:

$$\frac{\omega'^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)(\omega'^2-\omega^2)}=\frac{A}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)}+\frac{B}{(\omega'^2-\omega^2)}$$

$$(\omega'^2-\omega^2)A-\omega'^2+((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)B=0$$

This must be true for all ω' so we can set $\omega' = \omega$ to pick out B:

$$B = \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

Plug this back in and solve for A:

$$A = \frac{\omega_0^4 - \omega'^2 \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

So that finally we have:

$$\frac{\omega'^2}{((\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2)(\omega'^2 - \omega^2)} = \frac{1}{((\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2)} \left[\frac{\omega_0^4 - \omega'^2 \omega^2}{(\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2} + \frac{\omega^2}{\omega'^2 - \omega^2} \right]$$

We can further decompose the first fraction in the brackets:

$$\frac{\omega_0^4 - \omega'^2 \omega^2}{(\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2} = \frac{A}{(\omega_0^2 - \omega'^2 + i\gamma \omega')} + \frac{B}{(\omega_0^2 - \omega'^2 - i\gamma \omega')}$$

Multiply out, choose $B = \omega_0^2 - A$, and solve for A and B. This leads to the expansion:

$$\frac{\omega_0^4 - \omega'^2 \omega^2}{(\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2} = \left[\frac{\omega_0^2}{2} - \frac{\omega'(\omega_0^2 - \omega^2)}{2i\gamma} \right] \frac{1}{(\omega_0^2 - \omega'^2 + i\gamma \omega')} + \left[\frac{\omega_0^2}{2} + \frac{\omega'(\omega_0^2 - \omega^2)}{2i\gamma} \right] \frac{1}{(\omega_0^2 - \omega'^2 - i\gamma \omega')}$$

Plugging this back in, the whole integrand in completely expanded form becomes:

$$\frac{\omega'^2}{((\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2)(\omega'^2 - \omega^2)} = \frac{1}{((\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2)}$$
$$\times \left[\left[\frac{\omega_0^2}{2} - \frac{\omega'(\omega_0^2 - \omega^2)}{2i\gamma} \right] \frac{1}{(\omega_0^2 - \omega'^2 + i\gamma \omega')} + \left[\frac{\omega_0^2}{2} + \frac{\omega'(\omega_0^2 - \omega^2)}{2i\gamma} \right] \frac{1}{(\omega_0^2 - \omega'^2 - i\gamma \omega')} + \frac{\omega^2}{\omega'^2 - \omega^2} \right]$$

This may seem like a lot of work that just leads to a less useful form, but we must do this to solve the integral analytically.

$$\Re(\epsilon(\omega)/\epsilon_0) = 1 + \frac{2\lambda\gamma}{\pi} \frac{1}{((\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2)} \left[-\frac{\omega_0^2}{2} P \int_0^\infty \frac{1}{(\omega'^2 - \omega_0^2 - i\gamma \omega')} d\omega' \right.$$
$$+ \frac{(\omega_0^2 - \omega^2)}{2i\gamma} P \int_0^\infty \frac{\omega'}{\omega'^2 - \omega_0^2 - i\gamma \omega'} d\omega' - \frac{\omega_0^2}{2} P \int_0^\infty \frac{1}{\omega'^2 - \omega_0^2 + i\gamma \omega'} d\omega'$$
$$\left. - \frac{(\omega_0^2 - \omega^2)}{2i\gamma} P \int_0^\infty \frac{\omega'}{\omega'^2 - \omega_0^2 + i\gamma \omega'} d\omega' + \omega^2 P \int_0^\infty \frac{1}{\omega'^2 - \omega^2} d\omega' \right]$$

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2\lambda\gamma}{\pi}\frac{1}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}[$$

$$-\frac{\omega_0^2}{2}\left[\frac{1}{2\sqrt{\omega_0^2+i\gamma\omega'}}\ln\left(\frac{\omega'-\sqrt{\omega_0^2+i\gamma\omega'}}{\omega'+\sqrt{\omega_0^2+i\gamma\omega'}}\right)+\frac{1}{2\sqrt{\omega_0^2-i\gamma\omega'}}\ln\left(\frac{\omega'-\sqrt{\omega_0^2-i\gamma\omega'}}{\omega'+\sqrt{\omega_0^2-i\gamma\omega'}}\right)\right]_0^\infty$$

$$+\frac{(\omega_0^2-\omega^2)}{2i\gamma}\frac{1}{2}\left[\ln\left(\omega'^2-\omega_0^2-i\gamma\omega'\right)-\ln\left(\omega'^2-\omega_0^2+i\gamma\omega'\right)\right]_0^\infty]$$

Now we must be careful and expand out the logarithm of a complex number according to:

$$\ln(z)=\ln(|z|)+i\,Arg(z)$$

while evaluating case by case to make sure the answer ends up in the right quadrant of the complex plane. Several integrals drop out.

$$\Re(\epsilon(\omega)/\epsilon_0)=1+\frac{2\lambda\gamma}{\pi}\frac{1}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}\frac{(\omega_0^2-\omega^2)}{2i\gamma}\frac{1}{2}\left[i\left(\pi+\tan^{-1}\left(\frac{-\gamma\omega'}{\omega'^2-\omega_0^2}\right)\right)-i\left(-\pi+\tan^{-1}\left(\frac{\gamma\omega'}{\omega'^2-\omega_0^2}\right)\right)\right]_0^\infty$$

$$\boxed{\Re(\epsilon(\omega)/\epsilon_0)=1+\lambda\frac{(\omega_0^2-\omega^2)}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}}$$

This is just the result you get if you use the harmonic model with one resonant frequency and split up the dielectric constant into real and imaginary parts, where $\lambda=(N\,e^2\,f_0)/(\epsilon_0 m)$