Jackson 7.1 Homework Problem Solution

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PROBLEM:

For each set of Stokes parameters given below deduce the amplitude of the electric field, up to an overall phase, in both linear polarization and circular polarization bases and make an accurate drawing similar to Fig. 7.4 showing the lengths of the axes of one of the ellipses and its orientation.

(a) $s_0 = 3$, $s_1 = -1$, $s_2 = 2$, $s_3 = -2$

(b) $s_0 = 25$, $s_1 = 0$, $s_2 = 24$, $s_3 = 7$

SOLUTION:

As discussed in the notes, the Stokes parameters are defined in a linear polarization basis according to:

$$s_0 = |E_1|^2 + |E_2|^2$$

$$s_1 = |E_1|^2 - |E_2|^2$$

$$s_2 = 2\,\Re(E_1^* E_2)$$

$$s_3 = 2\,\Im(E_1^* E_2)$$

The Jones vector elements are just the magnitude and phase of each component, so let us rewrite these definitions explicitly:

$$s_0 = |E_1|^2 + |E_2|^2$$

$$s_1 = |E_1|^2 - |E_2|^2$$

$$s_2 = 2|E_1||E_2|\cos(\theta_2 - \theta_1)$$

$$s_3 = 2|E_1||E_2|\sin(\theta_2 - \theta_1)$$

Now invert these equations to solve for the Jones vector elements:

$$|E_1| = \sqrt{\frac{s_0 + s_1}{2}}$$

$$|E_2| = \sqrt{\frac{s_0 - s_1}{2}}$$

$$\theta_2-\theta_1=\cos^{-1}\left[\frac{s_2}{\sqrt{s_0^2-s_1^2}}\right] \quad \text{or} \quad \theta_2-\theta_1=\sin^{-1}\left[\frac{s_3}{\sqrt{s_0^2-s_1^2}}\right]$$

Note that the four Stokes parameters are not independent, so that we can only know the difference of the phase factors. The last piece of information, the overall phase factor of the whole system, is typically not as important because it depends on the choice of origin which can be anything:

$$\mathbf{E}_0=\begin{bmatrix}|E_1|e^{i\theta_1}\\ |E_2|e^{i\theta_2}\end{bmatrix}=e^{i\theta_1}\begin{bmatrix}|E_1|\\ |E_2|e^{i(\theta_2-\theta_1)}\end{bmatrix}$$

Similarly, the Stokes parameters are defined in a circular polarization basis according to:

$$s_0=|E_+|^2+|E_-|^2$$
$$s_1=2|E_+||E_-|\cos(\theta_- - \theta_+)$$
$$s_2=2|E_+||E_-|\sin(\theta_- - \theta_+)$$
$$s_3=|E_+|^2-|E_-|^2$$

Now invert these equations:

$$|E_+|=\sqrt{\frac{s_0+s_3}{2}}$$

$$|E_-|=\sqrt{\frac{s_0-s_3}{2}}$$

$$\theta_- - \theta_+=\cos^{-1}\left[\frac{s_1}{\sqrt{s_0^2-s_3^2}}\right] \quad \text{or} \quad \theta_- - \theta_+=\sin^{-1}\left[\frac{s_2}{\sqrt{s_0^2-s_3^2}}\right]$$

In Summary:

$$\boxed{\mathbf{E}_{0,\,\mathrm{lin}}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{s_0+s_1}\\ \dfrac{s_2+i\,s_3}{\sqrt{s_0+s_1}}\end{bmatrix}} \quad \text{and} \quad \boxed{S=\begin{bmatrix}|E_1|^2+|E_2|^2\\ |E_1|^2-|E_2|^2\\ 2|E_1||E_2|\cos(\theta_2-\theta_1)\\ 2|E_1||E_2|\sin(\theta_2-\theta_1)\end{bmatrix}}$$

$$\boxed{\mathbf{E}_{0,\,\mathrm{circ}}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{s_0+s_3}\\ \dfrac{s_1+i\,s_2}{\sqrt{s_0+s_3}}\end{bmatrix}} \quad \text{and} \quad \boxed{S=\begin{bmatrix}|E_+|^2+|E_-|^2\\ 2|E_+||E_-|\cos(\theta_- - \theta_+)\\ 2|E_+||E_-|\sin(\theta_- - \theta_+)\\ |E_+|^2-|E_-|^2\end{bmatrix}}$$

(a) $\mathbf{E}_{0,\,\mathrm{lin}}=\begin{bmatrix}1\\1-i\end{bmatrix}$ and $\mathbf{E}_{0,\,\mathrm{circ}}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1+2i\end{bmatrix}$

or $\mathbf{E}_{0,\,\mathrm{lin}}=\begin{bmatrix}1\\\sqrt{2}\,e^{i7\pi/4}\end{bmatrix}$ and $\mathbf{E}_{0,\,\mathrm{circ}}=\begin{bmatrix}1/\sqrt{2}\\\sqrt{5/2}\,e^{i(\pi+\tan^{-1}(-2))}\end{bmatrix}$

(b) $\mathbf{E}_{0,\,\mathrm{lin}}=\frac{1}{\sqrt{2}}\begin{bmatrix}5\\\dfrac{24+i7}{5}\end{bmatrix}$ and $\mathbf{E}_{0,\,\mathrm{circ}}=\begin{bmatrix}4\\3i\end{bmatrix}$

or $\mathbf{E}_{0,\,\mathrm{lin}}=\begin{bmatrix}5/\sqrt{2}\\5/\sqrt{2}\,e^{i\tan^{-1}(7/24)}\end{bmatrix}$ and $\mathbf{E}_{0,\,\mathrm{circ}}=\begin{bmatrix}4\\3\,e^{i\pi/2}\end{bmatrix}$