Jackson 7.19 Homework Problem Solution

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PROBLEM:

An approximately monochromatic plane wave packet in one dimension has the instantaneous form, $u(x,0)=f(x)e^{ik_0x}$, with $f(x)$ the modulation envelope. For each of the forms $f(x)$ below, calculate the wave-number spectrum $|A(k)|^2$ of the packet, sketch $|u(x,0)|^2$ and $|A(k)|^2$, evaluate explicitly the rms deviations from the means Δx and Δk (defined in terms of the intensities $|u(x,0)|^2$ and $|A(k)|^2$), and test inequality (7.82).

(a) $f(x)=N\,e^{-\alpha|x|/2}$

(b) $f(x)=N\,e^{-\alpha^2x^2/4}$

(c) $f(x)=\begin{bmatrix} N(1-\alpha|x|) \;\text{ for }\; \alpha|x|<1 \\ 0 \;\text{ for }\; \alpha|x|>1 \end{bmatrix}$

(d) $f(x)=\begin{bmatrix} N \;\text{ for }\; |x|<a \\ 0 \;\text{ for }\; |x|>a \end{bmatrix}$

SOLUTION:

First note the definition of rms for a wave packet: $\Delta x=\sqrt{\dfrac{\int_{-\infty}^{\infty} x^2[f(x)]^2\,dx}{\int_{-\infty}^{\infty}[f(x)]^2\,dx}}$

(a) For $u(x,0)=N\,e^{ik_0x-\alpha|x|/2}$ the wave-number spectrum is:

$$A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,0)e^{-ikx}\,dx$$

$$A(k)=\frac{1}{\sqrt{2\pi}}N\int_{-\infty}^{\infty}e^{-ikx+ik_0x-\alpha|x|/2}\,dx$$

$$A(k)=\frac{1}{\sqrt{2\pi}}2N\left[\int_0^{\infty}\cos((k-k_0)x)e^{-\alpha x/2}\,dx\right]$$

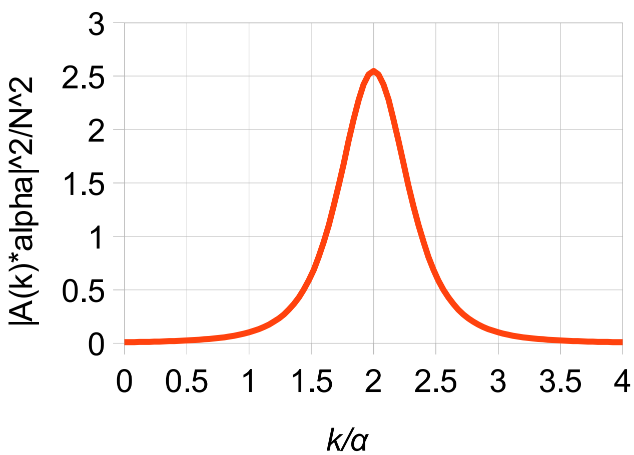
$$A(k)=\frac{1}{\sqrt{2\pi}}2N\left[e^{-\alpha x/2}\frac{-\frac{\alpha}{2}\cos((k-k_0)x)+(k-k_0)\sin((k-k_0)x)}{\alpha^2/4+(k-k_0)^2}\right]_0^{\infty}$$

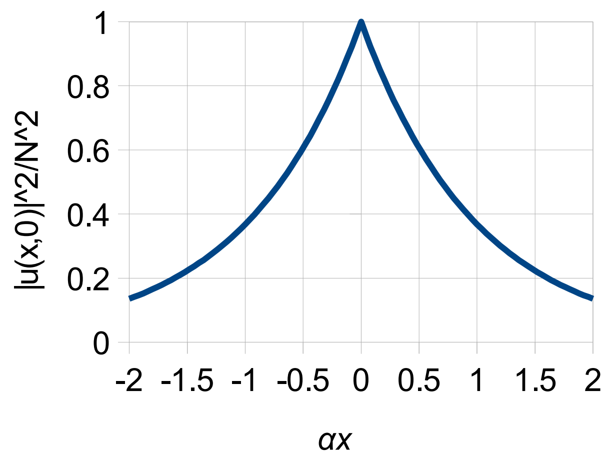
$$A(k)=\frac{1}{\sqrt{2\pi}}\left[\frac{N\alpha}{\alpha^2/4+(k-k_0)^2}\right]$$

The modulus squared is therefore found (in a form normalized for plotting):

$$\boxed{\frac{|A(k)|^2\alpha^2}{N^2}=\frac{1}{2\pi}\left[\frac{1}{\frac{1}{4}+\left(\frac{k}{\alpha}-\frac{k_0}{\alpha}\right)^2}\right]^2}$$ This is the wave-number spectrum of: $$\boxed{\frac{|u(x,0)|^2}{N^2}=e^{-|\alpha x|}}$$

We can plot this spectrum (with $k_0 = 2\alpha$) and its corresponding wave packet:





The spread of the wave packet and its spectrum can now be found:

$$\Delta x=\sqrt{\frac{\int_{-\infty}^{\infty}x^2e^{-\alpha|x|}dx}{\int_{-\infty}^{\infty}e^{-\alpha|x|}dx}}$$

$$\boxed{\Delta x=\frac{\sqrt{2}}{\alpha}}$$

$$\Delta k=\sqrt{\frac{\int_{-\infty}^{\infty}k^2\left[\frac{1}{\alpha^2/4+k^2}\right]^2dk}{\int_{-\infty}^{\infty}\left[\frac{1}{\alpha^2/4+k^2}\right]^2dk}}$$

$$\boxed{\Delta k=\frac{\alpha}{2}}$$

$\Delta x\Delta k=\frac{\sqrt{2}}{2}$ for this case, which obeys $\Delta x\Delta k\geq\frac{1}{2}$

(b) For $u(x,0)=N\,e^{i k_0 x-\alpha^2 x^2/4}$ the wave-number spectrum is:

$$A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{-ikx}\,dx$$

$$A(k)=\frac{1}{\sqrt{2\pi}}N\int_{-\infty}^{\infty} e^{i k_0 x-ikx-\alpha^2 x^2/4}\,dx$$

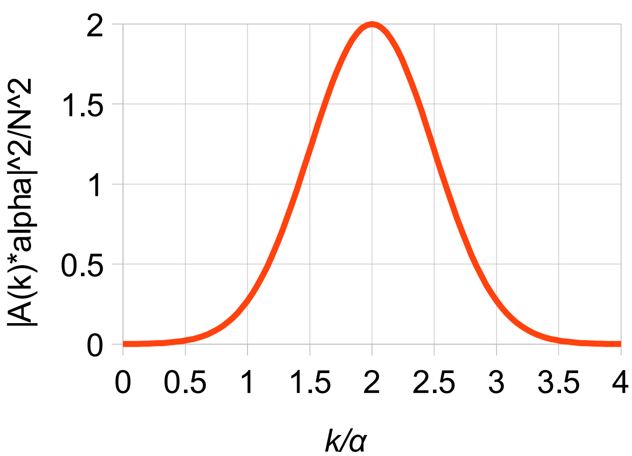
$$A(k)=\frac{2}{\sqrt{2\pi}}N\int_{0}^{\infty} \cos((k-k_0)x)e^{-\alpha^2 x^2/4}\,dx$$

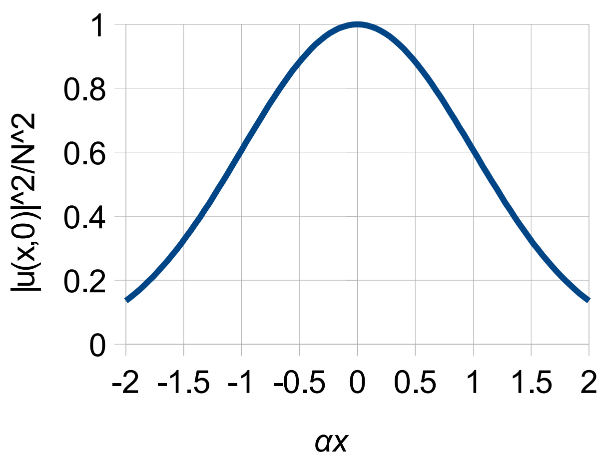
$$A(k)=N\sqrt{\frac{2}{\alpha^2}}e^{-(k-k_0)^2/\alpha^2}$$

The modulus squared is therefore found (in a form normalized for plotting):

$$\boxed{\frac{|A(k)|^2\alpha^2}{N^2}=2e^{-2\left(\frac{k}{\alpha}-\frac{k_0}{\alpha}\right)^2}}$$ This is the wave-number spectrum of: $$\boxed{\frac{|u(x,0)|^2}{N^2}=e^{-(\alpha x)^2/2}}$$

We can plot this spectrum (with $k_0 = 2\alpha$) and its corresponding wave packet:





The spread of the wave packet and its spectrum can now be found:

$$\Delta x=\sqrt{\frac{\int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2/2}\,dx}{\int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2}\,dx}}$$

$$\boxed{\Delta x=\frac{1}{\alpha}}$$

$$\Delta k = \sqrt{\frac{\int_{-\infty}^{\infty} k^2 e^{-2k^2/\alpha^2} dk}{\int_{-\infty}^{\infty} e^{-2k^2/\alpha^2} dk}}$$

$$\boxed{\Delta k = \frac{\alpha}{2}}$$

$\Delta x \Delta k = \frac{1}{2}$ for this case, which still obeys $\Delta x \Delta k \geq \frac{1}{2}$

Note that this wave packet reaches the minimum uncertainty possible. This means that this wave packet has maximum smoothness.

(c) For the case of $u(x,0) = N(1-\alpha|x|)e^{ik_0 x}$ for $\alpha|x|<1, 0$ otherwise, the wave-number spectrum is:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx$$

$$A(k) = \frac{2N}{\sqrt{2\pi}} \int_{0}^{1/\alpha} (1-\alpha x) \cos((k-k_0)x) dx$$

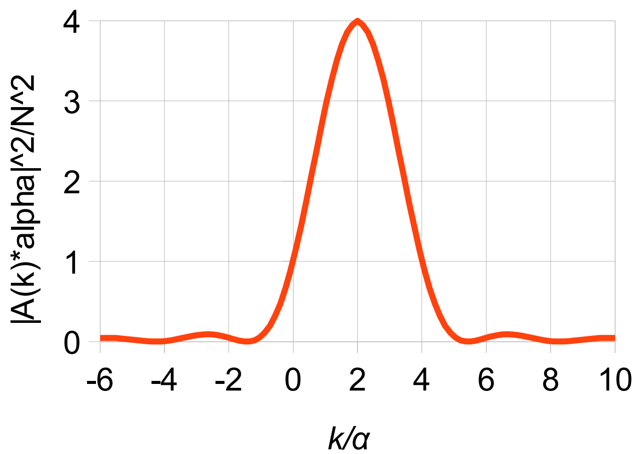
$$A(k) = \frac{2N}{\alpha\sqrt{2\pi}} \frac{1}{(k/\alpha - k_0/\alpha)^2} [2(k/\alpha - k_0/\alpha)\sin(k/\alpha - k_0/\alpha) + 1 - \cos(k/\alpha - k_0/\alpha)]$$

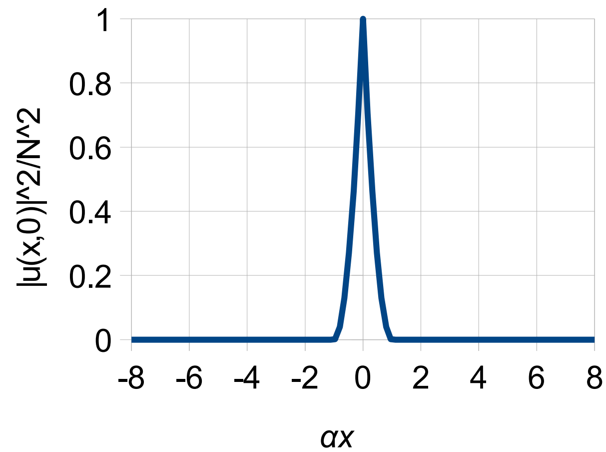
The modulus squared is therefore found (in a form normalized for plotting):

$$\boxed{\frac{|A(k)|^2 \alpha^2}{N^2} = \frac{2}{\pi} \frac{1}{\left(\frac{k}{\alpha} - \frac{k_0}{\alpha}\right)^4} \left[2\left(\frac{k}{\alpha} - \frac{k_0}{\alpha}\right)\sin\left(\frac{k}{\alpha} - \frac{k_0}{\alpha}\right) + 1 - \cos\left(\frac{k}{\alpha} - \frac{k_0}{\alpha}\right)\right]^2}$$

This is the wave-number spectrum of: $\boxed{\frac{|u(x,0)|^2}{N^2} = (1-|\alpha x|)^2 \text{ for } |\alpha x|<1, 0 \text{ otherwise}}$

We can plot this spectrum (with $k_0 = 2\alpha$) and its corresponding wave packet:





Note that the discontinuous nature of the slope of the waveform in coordinate space leads to ringing in wavenumber space. The spread of the wave packet and its spectrum can now be found:

$$\Delta x = \frac{1}{\alpha}\sqrt{\frac{\int_0^1 x^2(1-x)^2\,dx}{\int_0^1 (1-x)^2\,dx}}$$

$$\boxed{\Delta x = \frac{1}{10\alpha}}$$

$$\Delta k = \alpha\sqrt{\frac{\int_0^\infty \frac{1}{k^2}[2k\sin k + 1 - \cos k]^2\,dk}{\int_0^\infty \frac{1}{k^4}[2k\sin k + 1 - \cos k]^2\,dk}}$$

Note that the highest power term in the numerator is $4\int_0^\infty \sin^2 k\,dk$ which obviously diverges, so that:

$$\boxed{\Delta k = \infty}$$

$\Delta x\,\Delta k = \infty$ for this case, which certainly obeys $\Delta x\,\Delta k \geq \frac{1}{2}$

(d) For $u(x,0)=N e^{i k_0 x}$ for $|x|<a$, 0 otherwise , the wave-number spectrum is:

$$A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0) e^{-ikx}\,dx$$

$$A(k)=\frac{1}{\sqrt{2\pi}} N \int_{-a}^{a} e^{-i(k-k_0)x}\,dx$$

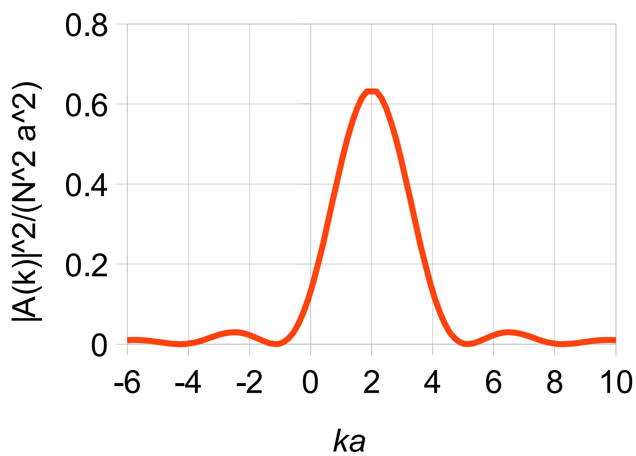
$$A(k)=\frac{2}{\sqrt{2\pi}} N \frac{\sin((k-k_0)a)}{(k-k_0)}$$

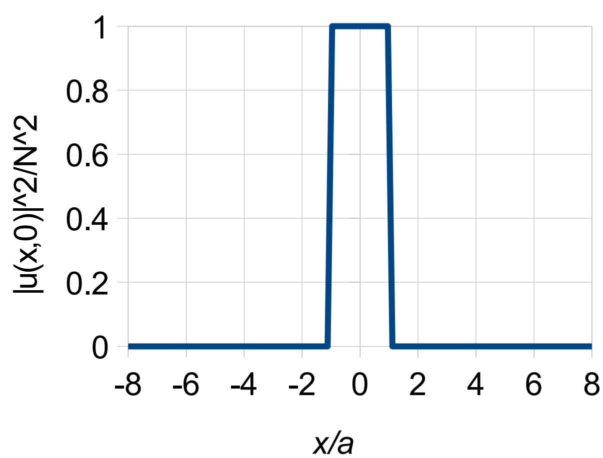
The modulus squared is therefore found (in a form normalized for plotting):

$$\boxed{\frac{|A(k)|^2}{N^2 a^2}=\frac{2}{\pi}\frac{\sin^2(ka-k_0 a)}{(ka-k_0 a)^2}}$$

This is the wave-number spectrum of: $\boxed{\frac{|u(x,0)|^2}{N^2}=1 \text{ for } |x/a|<1, 0 \text{ otherwise}}$

We can plot this spectrum (with $k_0 = 2/a$) and its corresponding wave packet:





The waveform itself is discontinuous, so there is significant ringing in wavenumber space. The spread of the wave packet and its spectrum can now be found:

$$\Delta x=\sqrt{\frac{\int_{-a}^{a} x^2\,dx}{\int_{-a}^{a} dx}}$$

$$\boxed{\Delta x=\frac{a}{\sqrt{3}}}$$

$$\Delta k = \sqrt{\frac{\int_{-\infty}^{\infty} \sin^2(k a) dk}{\int_{-\infty}^{\infty} \frac{\sin^2(k a)}{k^2} dk}}$$

$$\boxed{\Delta k = \infty}$$

$\Delta x \Delta k = \infty$ for this case, which certainly obeys $\Delta x \Delta k \geq \frac{1}{2}$