Jackson 7.12 Homework Problem Solution

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PROBLEM:

The time dependence of electrical disturbances in good conductors is governed by the frequency-dependent conductivity (7.58). Consider longitudinal electric fields in a conductor, using Ohm's law, the continuity equation, and the differential form of Coulomb's law.

(a) Show that the time-Fourier-transformed charge density satisfies the equation

$$[\sigma(\omega) - i\omega\epsilon_0]\rho(\mathbf{x},\omega) = 0$$

(b) Using the representation $\sigma(\omega) = \sigma_0/(1 - i\omega\tau)$, where $\sigma_0 = \epsilon_0\omega_p^2\tau$ and τ is a damping time, show that in the approximation $\omega_p\tau >> 1$ any initial disturbance will oscillate with the plasma frequency and decay in amplitude with a decay constant $\lambda = 1/2\tau$. Note that if you use $\sigma(\omega) = \sigma(0) = \sigma_0$ in part a, you will find no oscillations and extremely rapid damping with the (wrong) decay constant $\lambda_w = \sigma_0/\varepsilon_0$.

SOLUTION:

(a) We are assuming that the conductivity dominates so that $\epsilon = \epsilon_0$ and therefore $\mathbf{D} = \epsilon_0\mathbf{E}$. The time Fourier transforms are defined according to:

$$\rho(\omega) = \frac{1}{\sqrt{2\pi}}\int \rho(t) e^{i\omega t} dt \quad \text{where} \quad \rho(t) = \frac{1}{\sqrt{2\pi}}\int \rho(\omega) e^{-i\omega t} d\omega$$

$$\mathbf{J}(\omega) = \frac{1}{\sqrt{2\pi}}\int \mathbf{J}(t) e^{i\omega t} dt \quad \text{where} \quad \mathbf{J}(t) = \frac{1}{\sqrt{2\pi}}\int \mathbf{J}(\omega) e^{-i\omega t} d\omega$$

$$\mathbf{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int \mathbf{E}(t) e^{i\omega t} dt \quad \text{where} \quad \mathbf{E}(t) = \frac{1}{\sqrt{2\pi}}\int \mathbf{E}(\omega) e^{-i\omega t} d\omega$$

The charge continuity equation states that: $\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$

Fourier transform the continuity equation to get it into frequency space:

$$\nabla\cdot\mathbf{J}(\omega) = i\omega\rho(\omega)$$

Ohm's law states that $\mathbf{J}(\omega) = \sigma(\omega)\mathbf{E}(\omega)$. Inserting Ohm's law into the continuity equation in frequency space, we find:

$$\sigma(\omega)\nabla\cdot\mathbf{E}(\omega) = i\omega\rho(\omega)$$

We have assumed the conductor material is spatially uniform in order to take σ out of the divergence operator. Now on the right we have the divergence of the electric field, which reminds us of Coulomb's law in differential form: $\nabla\cdot\mathbf{E}(\omega)=\rho(\omega)/\epsilon_0$. (Because $\epsilon=\epsilon_0$, and there are no time operators, Coulomb's law looks the same in frequency space and time space.) Insert Coulomb's law in the equation above to find:

$$\boxed{[\sigma(\omega)-i\omega\,\epsilon_0]\rho(\omega)=0}$$

(b) Now use the representation $\sigma(\omega)=\sigma_0/(1-i\omega\tau)$, where $\sigma_0=\epsilon_0\omega_p^2\tau$ and τ is a damping time.

$$\boxed{[\sigma(\omega)-i\omega\,\epsilon_0]\rho(\omega)=0}$$

$$\left[\frac{\omega_p^2\tau}{1-i\omega\tau}-i\omega\right]\rho(\omega)=0$$

If the charge density is to exist (be non-zero), then the factor in brackets must vanish.

$$\left[\frac{\omega_p^2\tau}{1-i\omega\tau}-i\omega\right]=0$$

$$\tau\omega^2+i\omega-\omega_p^2\tau=0$$

$$\omega=\frac{-i\pm\sqrt{4\tau^2\omega_p^2-1}}{2\tau}$$

In the approximation $\omega_p\tau >> 1$

$$\omega=\pm\omega_p-i\frac{1}{2\tau}$$

This tells us that $\rho(\omega)=\begin{bmatrix}\rho_{0,+}\ \text{if}\ \omega=+\omega_p-i/2\tau\\ \rho_{0,-}\ \text{if}\ \omega=-\omega_p-i/2\tau\\ 0\ \text{for all other frequencies}\end{bmatrix}$

Plugging this into the definition of the charge density:

$$\rho(t)=\frac{1}{\sqrt{2\pi}}\int\rho(\omega)e^{-i\omega t}\,d\omega$$

$$\rho(t)=\rho_{0,+}e^{-i(\omega_p-i/2\tau)t}+\rho_{0,-}e^{-i(-\omega_p-i/2\tau)t}$$

$$\boxed{\rho(t)=[\rho_{0,+}e^{-i\omega_p t}+\rho_{0,-}e^{i\omega_p t}]e^{-t/2\tau}}$$

This tells us that an initial charge distribution will oscillate at the plasma frequency and decay with decay constant $1/2\tau$.