Jackson 6.9 Homework Problem Solution

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PROBLEM:

Discuss the conservation of energy and linear momentum for a macroscopic system of sources and electromagnetic fields in a uniform, isotropic medium described by a permittivity ε and a permeability μ. Show that in a straightforward calculation the energy density, Poynting vector, field-momentum density, and Maxwell stress tensor are given by the Minkowski expressions:

$$u=\frac{1}{2}(\epsilon E^2+\mu H^2)$$

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}$$

$$\mathbf{g}=\mu\epsilon\mathbf{E}\times\mathbf{H}$$

$$T_{ij}=[\epsilon E_i E_j+\mu H_i H_j-\frac{1}{2}\delta_{ij}(\epsilon E^2+\mu H^2)]$$

What modifications are made if ε and μ are functions of position?

SOLUTION:

As derived in class, the energy density and energy flux density in linear, low-dispersion, low-loss materials are given by:

$$u=\frac{1}{2}(\mathbf{H}\cdot\mathbf{B}+\mathbf{E}\cdot\mathbf{D}) \quad \text{and} \quad \mathbf{S}=\mathbf{E}\times\mathbf{H}$$

For linear material, $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \varepsilon\mathbf{E}$ so these become:

$$\boxed{u=\frac{1}{2}(\epsilon E^2+\mu H^2)} \quad \text{and} \quad \boxed{\mathbf{S}=\mathbf{E}\times\mathbf{H}}$$

The field-momentum density is given by:

$$\mathbf{g}=\frac{\mathbf{S}}{v^2}$$

In linear materials, the speed of the waves are given by $v=\frac{1}{\sqrt{\epsilon\mu}}$ so that

$$\boxed{\mathbf{g}=\mu\epsilon\mathbf{E}\times\mathbf{H}}$$

The Maxwell stress tensor describes the electromagnetic momentum flux and is given by:

$$T_{ij} = [\epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\delta_{ij}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)]$$

For linear materials, replace the permeability and permittivity of free space with the material's values:

$$T_{ij} = [\epsilon E_i E_j + \frac{1}{\mu} B_i B_j - \frac{1}{2}\delta_{ij}(\epsilon E^2 + \frac{1}{\mu} B^2)]$$

Switch out **B** for **H** using **B** = μ**H**

$$\boxed{T_{ij} = [\epsilon E_i E_j + \mu H_i H_j - \frac{1}{2}\delta_{ij}(\epsilon E^2 + \mu H^2)]}$$

Seeing as all of these equations are position-dependent densities, they automatically take into account the possibility of position-dependent permeabilities and permittivities. We do not need to change anything if ε and μ are functions of position. This is the advantage of working with densities instead of total values.