Jackson 6.8 Homework Problem Solution

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PROBLEM:

A dielectric sphere of dielectric constant ε and radius a is located at the origin. There is a uniform electric field E_0 in the x direction. The sphere rotates with an angular velocity ω about the z axis. Show that there is a magnetic field $\mathbf{H}=-\nabla\Phi_M$, where

$$\Phi_M=\frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0\omega\left(\frac{a}{r_>}\right)^5 xz$$

where $r_>$ is the larger of r and a. The motion is non-relativistic. You may use the results of Section 4.4 for the dielectric sphere in an applied field.

SOLUTION:

The basic idea is that the applied electric field induces a bound surface charges on the sphere, which become bound currents due to the rotation of the sphere. These currents then give rise to magnetic fields.

From Section 4.4, we know that a uniform electric field applied in the x direction to a dielectric sphere induces a bound surface charge:

$$\sigma_{\text{pol}}=3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0\hat{\mathbf{r}}\cdot\hat{\mathbf{x}}$$

$$\sigma_{\text{pol}}=3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0\sin\theta\cos\phi \quad \text{in spherical coordinates}$$

$$\boxed{\sigma_{\text{pol}}=3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0\frac{x}{a}} \quad \text{in rectangular coordinates}$$

Because this charge is spinning on the surface of a sphere of radius a at angular velocity ω we have a bound surface current density:

$$\mathbf{K}_M=\sigma\,\mathbf{v}$$

$$\mathbf{K}_M=\sigma\,\boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{K}_M=\sigma(\omega\hat{\mathbf{z}})\times(a\hat{\mathbf{r}})$$

$$\mathbf{K}_M=\sigma\,\omega\,a\sin\theta\,\hat{\boldsymbol{\phi}}$$

$$\boxed{\mathbf{K}_M = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega x \sin\theta\,\hat{\boldsymbol{\phi}}}$$

Now if the problem had asked for the fields, we could have integrated over this current directly. But the problem instead asks for the magnetic potential. The magnetic potential is an integral over effective magnetic charges, so we must transform our effective magnetic current to effective magnetic charges.

$$\mathbf{M}\times\hat{\mathbf{r}} = \mathbf{K}_M$$

$$(M\,\hat{\mathbf{z}})\times\hat{\mathbf{r}} = \mathbf{K}_M$$

$$M \sin\theta\,\hat{\boldsymbol{\phi}} = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega\, x \sin\theta\,\hat{\boldsymbol{\phi}}$$

$$\boxed{\mathbf{M} = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega\, x\,\hat{\mathbf{z}}}$$

Even though this magnetization is not constant, it is still without divergence (you can check this easily) so that there is no associated volume effective magnetization charge, only a surface one:

$$\sigma_M = \hat{\mathbf{r}}\cdot\mathbf{M}$$

$$\boxed{\sigma_M = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \frac{\omega}{a} x z}$$

$$\Phi_M = \frac{1}{4\pi}\int \frac{\sigma_M}{|\mathbf{x}-\mathbf{x}'|} da'$$

Expand the denominator into spherical harmonics using:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$\Phi_M = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \frac{\omega}{a}\int_0^{\pi}\int_0^{2\pi} a^2 \sin\theta'\cos\theta'\cos\phi' \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)\, a^2 \sin\theta'\, d\theta'\, d\phi'$$

$$\Phi_M = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega\, a^3 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi)\int_0^{\pi}\int_0^{2\pi} \sin\theta'\cos\theta'\cos\phi'\, Y^*_{lm}(\theta',\phi')\sin\theta'\, d\theta'\, d\phi'$$

Use $\sin\theta\cos\theta\cos\phi = \sqrt{\frac{2\pi}{15}}\left(Y_{2-1} - Y_{21}\right)$

$$\Phi_M = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega a^3 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi)\int_0^{\pi}\int_0^{2\pi}\sqrt{\frac{2\pi}{15}}(Y_{2-1}-Y_{21})Y_{lm}^{*}\sin\theta' d\theta' d\phi'$$

$$\Phi_M = 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega a^3 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi)\sqrt{\frac{2\pi}{15}}[\delta_{l,2}\delta_{m,-1}-\delta_{l,2}\delta_{m,1}]$$

$$\Phi_M = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega a^3 \frac{r_<^2}{r_>^3}\frac{xz}{r^2}$$

Break this up into cases

$$\Phi_M = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega\, xz \quad \text{if } r < a$$

$$\Phi_M = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega \frac{a^5}{r^5} xz \quad \text{if } r > a$$

Recombine:

$$\boxed{\Phi_M = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega\left(\frac{a}{r_>}\right)^5 xz}$$