Jackson 6.4 Homework Problem Solution

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PROBLEM:

A uniformly magnetized and conducting sphere of radius R and total magnetic moment $m = 4\pi MR^3/3$ rotates about its magnetization axis with angular speed ω. In the steady state no current flows in the conductor. The motion is non-relativistic; the sphere has no excess charge on it.

(a) By considering Ohm's law in the moving conductor, show that the motion induces an electric field and a uniform volume charge density in the conductor, $\rho = -m\omega/\pi c^2R^3$.

(b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to show that the lowest possible electric multipolarity is quadrupole. Show that only a quadrupole field exists outside and that the quadrupole moment tensor has non-vanishing components, $Q_{33} = -4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = -Q_{11}/2$

(c) By considering the radial electric fields inside and outside the sphere, show that the necessary surface-charge density $\sigma(\theta)$ is

$$\sigma(\theta)=\frac{1}{4\pi R^2}\frac{4m\omega}{3c^2}\left[1-\frac{5}{2}P_2(\cos\theta)\right]$$

(d) The rotating sphere serves as a unipolar induction device if a stationary circuit is attached by a slip ring to the pole and a sliding contact to the equator. Show that the line integral of the electric field from the equator contact to the pole contact (by any path) is $E_l = \mu_0 m\omega/4\pi R$.

SOLUTION:

(a) Place the sphere at the origin and align the spin axis with the z axis. Because the total magnetic moment $\mathbf{m}$ is the spatial integral of the magnetization $\mathbf{M}$ and the sphere is uniformly magnetized, so that

$$\mathbf{m}=\int \mathbf{M}(\mathbf{x})\,d\,\mathbf{x}=\mathbf{M}\,V=\mathbf{M}\frac{4}{3}\pi R^3$$

it becomes obvious that the the magnetization is $\mathbf{M}=M\,\hat{\mathbf{z}}$. Because the motion is slow compared to the speed of light, we can make the assumption that electric field $\mathbf{D}$ varies very slowly with time so that we can assume:

$$\frac{\partial \mathbf{D}}{\partial t}=0$$

Additionally, the current is not defined in the laboratory frame, so that in this frame, Ampere's law becomes:

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\ \rightarrow\ \nabla\times\mathbf{H}=0$$

In the laboratory frame then, the problem reduces to the magnetostatics problem of finding the magnetic fields associated with a uniformly magnetized sphere. We already did this problem and found inside the sphere:

$$\mathbf{H}=-\frac{1}{3}M\,\hat{\mathbf{z}}\quad\text{and}\quad\mathbf{B}=\frac{2}{3}\mu_0 M\,\hat{\mathbf{z}}$$

In terms of the total magnetic moment of the sphere, (using $M=3\,m/4\pi\,R^3$) this becomes

$$\mathbf{H}=-\frac{m}{4\pi\,R^3}\hat{\mathbf{z}}\quad\text{and}\quad\mathbf{B}=\frac{\mu_0 m}{2\pi\,R^3}\hat{\mathbf{z}}$$

Current is always measured in the frame of reference where the conductor is at rest. This means that there is no current flowing in the spinning frame of reference **J** = 0. Because the motion is non-relativistic, we can use the Galilean transformation to relate the electric field in the rotating frame **E**' to the electric field **E** and magnetic field **B** in the lab frame:

$$\mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{B}$$

Ohm's Law states that **J** = σ**E**' in the rotating frame. Because there is no current **J** = 0, there must be no electric field in the rotating frame **E**' = 0 so that $0=\mathbf{E}+\mathbf{v}\times\mathbf{B}$ and

$$\mathbf{E}=-\mathbf{v}\times\mathbf{B}$$

$$\mathbf{E}=-(\boldsymbol{\omega}\times\mathbf{r})\times\mathbf{B}$$

$$\mathbf{E}=-\omega(\hat{\mathbf{z}}\times\mathbf{r})\times\mathbf{B}$$

$$\mathbf{E}=\frac{-\mu_0 m\omega}{2\pi\,R^3}(\hat{\mathbf{z}}\times\mathbf{r})\times\hat{\mathbf{z}}$$

$$\boxed{\mathbf{E}=\frac{-\mu_0 m\omega}{2\pi\,R^3}\rho\,\hat{\boldsymbol{\rho}}}$$ where we are now in cylindrical coordinates

It is straightforward to use Gauss's Law to find the volume charge density ρ (not to be confused with the cylindrical radial coordinate):

$$\rho=\epsilon_0\nabla\cdot\mathbf{E}$$

$$\rho=\epsilon_0\frac{\partial E_\rho}{\partial\rho}$$

$$\rho=-\frac{\epsilon_0\mu_0 m\,\omega}{2\pi\,R^3}$$

$$\boxed{\rho = -\frac{m\omega}{2\pi c^2 R^3}}$$

(b) Because the sphere is electrically neutral, there is no monopole electric field outside. Use symmetry arguments to show that the lowest possible electric multipolarity is quadrupole. Show that only a quadrupole field exists outside and that the quadrupole moment tensor has non-vanishing components, $Q_{33} = - 4m\omega R^2/3c^2$, $Q_{11} = Q_{22} = - Q_{11}/2$

No charge exists outside the sphere. We can thus make a multipole expansion of the electric field outside. It is obvious from the symmetry of the problem that the fields at some point (x, y, z) must equal the fields at the point (x, y, $-z$). Thus all the dipole and all other odd multipoles must be zero because they do not obey this property. The quadrupole must be the lowest non-vanishing term in the expansion. The higher multipole terms are found to vanish by a direct calculation.

The electric field inside in terms of spherical coordinates is:

$$\mathbf{E} = \frac{-\mu_0 m\omega}{2\pi R^3}(r\sin^2\theta\,\hat{\mathbf{r}} + r\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}})$$

After dropping the monopole term, the general solution outside is:

$$\mathbf{E} = -\nabla\left[\sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)\right]$$

$$\mathbf{E} = -\hat{\mathbf{r}}\frac{\partial}{\partial r}\left[\sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)\right] - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left[\sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)\right]$$

$$\mathbf{E} = -\hat{\mathbf{r}}\left[\sum_{l=1}^{\infty} B_l(-l-1) r^{-(l+2)} P_l(\cos\theta)\right] - \hat{\boldsymbol{\theta}}\left[-B_1 r^{-3}\sin\theta - B_2 r^{-4} 3\cos\theta\sin\theta + B_3 r^{-5}\left(\frac{-15}{2}\cos^2\theta\sin\theta + \frac{3}{2}\sin\theta\right) + \ldots\right.$$

Apply the boundary conditions. The tangential components of the **E** field must be continuous across $r = R$

$$\frac{-\mu_0 m\omega}{2\pi R^2}\sin\theta\cos\theta = B_1 R^{-3}\sin\theta + B_2 R^{-4} 3\cos\theta\sin\theta + B_3 R^{-5}\left(\frac{15}{2}\cos^2\theta\sin\theta - \frac{3}{2}\sin\theta\right) + \ldots$$

Due to orthogonality, it becomes obvious that only the $l = 2$ term will contribute

$$B_2 = \frac{-\mu_0 m\omega}{6\pi} R^2 \quad \text{and } B_l = 0 \text{ for } l \neq 2$$

$$\boxed{\mathbf{E} = \frac{-\mu_0 m\omega}{2\pi}\frac{R^2}{r^4}\left[\hat{\mathbf{r}}\,P_2(\cos\theta) + \hat{\boldsymbol{\theta}}\cos\theta\sin\theta\right]} \quad \text{outside}$$

We can get this in the form of a multipole expansion to try to pick out the multipole moments

$$E_r = \frac{-\mu_0 m\omega}{2\pi}\frac{R^2}{r^4}P_2(\cos\theta) \quad \text{and} \quad E_\theta = \frac{\mu_0 m\omega}{6\pi}\frac{R^2}{r^4}\frac{\partial}{\partial\theta}P_2(\cos\theta)$$

Comparison with $E_r = \frac{3}{5\epsilon_0}\frac{q_{20}}{r^4}\sqrt{\frac{5}{4\pi}}P_2(\cos\theta)$ reveals that

$$\boxed{q_{20} = \frac{-2}{3}\sqrt{\frac{5}{4\pi}}\frac{m\omega R^2}{c^2}}$$

Converting to Cartesian tensors using the definition we find:

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}}q_{20}$$

$\boxed{Q_{33} = \frac{-4}{3}\frac{m\omega R^2}{c^2}}$ We use the traceless nature and the symmetry to find $\boxed{Q_{11} = Q_{22} = \frac{-1}{2}Q_{33}}$

(c) By considering the radial electric fields inside and outside the sphere, show that the necessary surface-charge density *σ*(*θ*) is

$$\sigma(\theta) = \frac{1}{4\pi R^2}\frac{4m\omega}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]$$

We now consider the other boundary condition:

$$(\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n} = \sigma$$

There is no dielectric material present, so this reduces to:

$$(\mathbf{E}_2 - \mathbf{E}_1)\cdot\mathbf{n} = \frac{\sigma}{\epsilon_0}$$

We know the electric field inside and outside, so we can use this to find the free surface charge density:

$$\sigma = \left[\epsilon_0(E_r^{\text{out}} - E_r^{\text{in}})\right]_{r=R}$$

$$\sigma = \epsilon_0\left[\frac{-\mu_0 m\omega}{2\pi}\frac{R^2}{r^4}P_2(\cos\theta) + \frac{\mu_0 m\omega}{2\pi R^3}r\sin^2\theta\right]_{r=R}$$

$$\boxed{\sigma(\theta) = \frac{1}{4\pi R^2}\frac{4m\omega}{3c^2}\left[1 - \frac{5}{2}P_2(\cos\theta)\right]}$$

(d) The rotating sphere serves as a unipolar induction device if a stationary circuit is attached by a slip ring to the pole and a sliding contact to the equator. Show that the line integral of the electric field from the equator contact to the pole contact (by any path) is $E_l = \mu_0 m\omega/4\pi R$.

$$E_l = -\int_0^{\pi/2} \mathbf{E}\cdot d\mathbf{l}$$

$$E_l = -R\int_0^{\pi/2} E_\theta\, d\theta$$

$$E_l = \frac{\mu_0 m \omega}{2\pi R}\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

$$\boxed{E_l = \frac{\mu_0 m \omega}{4\pi R}}$$