Jackson 6.1 Homework Problem Solution

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PROBLEM:

In three dimensions the solution to the wave equation (6.32) for a point source in space and time (a light flash at $t' = 0$, $\mathbf{x}' = 0$) is a spherical shell disturbance of radius $R = ct$, namely the Green function $G^{(+)}$ (6.44). It may be initially surprising that in one or two dimensions, the disturbance possesses a "wake" even though the source is a "point" in space and time. The solutions for fewer dimensions than three can be found by superposition in the superfluous dimension(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

(a) Starting with the retarded solution to the three-dimensional wave equation (6.47), show that the source $f(\mathbf{x}', t') = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two spatial dimensions, produces a two-dimensional wave,

$$\Phi(x, y, t) = \frac{2c\,\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}$$

where $\rho^2 = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function [$\Theta(\xi) = 0$ (1) if $\xi < (>)\ 0$.]

(b) Show that a "sheet" source, equivalent to a point pulsed source at the origin in one space dimension, produces a one-dimensional wave proportional to

$$\Phi(x, t) = 2\pi c\,\Theta(ct - |x|)$$

SOLUTION:

The retarded solution to the three-dimensional wave equation is:

$$\Phi(\mathbf{x}, t) = \int \frac{[f(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

For a normalized flashing point source in three dimensions at an arbitrary point, the source is

$$f(\mathbf{x}', t') = \delta(x'-x_0)\delta(y'-y_0)\delta(z'-z_0)\delta(t'-t_0)$$

so that:

$$[f(\mathbf{x}', t')]_{\text{ret}} = \delta(x'-x_0)\delta(y'-y_0)\delta(z'-z_0)\delta(t-|\mathbf{x}-\mathbf{x}'|/c-t_0)$$

Plugging this into the solution, we find:

$$\Phi(\mathbf{x},t)=\int \frac{\delta(x'-x_0)\delta(y'-y_0)\delta(z'-z_0)\delta(t-|\mathbf{x}-\mathbf{x}'|/c-t_0)}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$\Phi(\mathbf{x},t)=\frac{\delta(t-|\mathbf{x}-\mathbf{x}_0|/c-t_0)}{|\mathbf{x}-\mathbf{x}_0|}$$

This is the retarded Green function and represents a thin spherical shell expanding outwards and decreasing radially in strength.

(a) A flashing line source on the z axis is given by: $f(\mathbf{x}', t') = \delta(x')\delta(y')\delta(t')$. The retarded form is therefore:

$$[f(\mathbf{x}',t')]_{\text{ret}}=\delta(x')\delta(y')\delta(t-|\mathbf{x}-\mathbf{x}'|/c)$$

Plugging this into the general three-dimensional solution, we find:

$$\Phi(\mathbf{x},t)=\int \frac{[f(\mathbf{x}',t')]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$\Phi(\mathbf{x},t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta(x')\delta(y')\delta(t-\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dx'dy'dz'$$

$$\Phi(\mathbf{x},t)=\int_{-\infty}^{\infty} \frac{\delta(t-\sqrt{\rho^2+(z-z')^2}/c)}{\sqrt{\rho^2+(z-z')^2}}dz'$$

We can't directly apply the Dirac delta to collapse the integral because the integral is with respect to z' and the Dirac delta's dependence on z' is not linear. We need to transform it so it is. We can use the property:

$$\delta(f(x))=\sum_i \frac{\delta(x-x_i)}{\left|\left(\frac{df}{dx}\right)_{x=x_i}\right|}$$ where x_i are the points where $f(x) = 0$

In our case $x = z'$, so that $f(x)=t-\sqrt{\rho^2+(z-z')^2}/c$ and $\frac{df}{dx}=\frac{1}{c}\frac{z-z'}{\sqrt{\rho^2+(z-z')^2}}$

The zeros of $f(x)$ are $z'=z\pm\sqrt{c^2t^2-\rho^2}$ but they only exist for $c^2t^2-\rho^2>0$

Putting all this together, we have:

$$\delta(t-\sqrt{\rho^2+(z-z')^2}/c)=\frac{\delta(z'-z-\sqrt{c^2t^2-\rho^2})}{\frac{1}{c}\frac{\sqrt{c^2t^2-\rho^2}}{\sqrt{\rho^2+(z-z')^2}}}+\frac{\delta(z'-z+\sqrt{c^2t^2-\rho^2})}{\frac{1}{c}\frac{\sqrt{c^2t^2-\rho^2}}{\sqrt{\rho^2+(z-z')^2}}}$$

$$\delta(t-\sqrt{\rho^2+(z-z')^2}/c)=c^2 t\frac{\delta(z'-z-\sqrt{c^2t^2-\rho^2})}{\sqrt{c^2t^2-\rho^2}}+c^2 t\frac{\delta(z'-z+\sqrt{c^2t^2-\rho^2})}{\sqrt{c^2t^2-\rho^2}}$$

Putting this into the integral we have:

$$\Phi(\mathbf{x},t)=\frac{2c}{\sqrt{c^2t^2-\rho^2}} \text{ if } c^2t^2-\rho^2>0 \text{ and zero otherwise}$$

or

$$\boxed{\Phi(x,y,t)=\frac{2c\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}}}$$

This represents a cylindrical expanding shell. Beyond the shell, the field is zero. On the shell, the field is a maximum (it is infinite, in fact, but that is just an artifact of having an infinitely-long line charge which is not physically realizable). Within the shell, the field is non-zero (there is a wake) but is quickly dieing down to zero. The wake is a result of the signal arriving from lateral points on the line charge that are far away from the observation points. Because they are farther away, their signal takes longer to reach the observation point and are therefore behind the main shell, and their signal strength is weaker.

(b) A flashing sheet source at $x = 0$ plane is given by:

$$f(\mathbf{x}',t')=\delta(x')\delta(t')$$

so that:

$$[f(\mathbf{x}',t')]_{\text{ret}}=\delta(x')\delta(t-|\mathbf{x}-\mathbf{x}'|/c)$$

Plugging this in the three-dimensional solution:

$$\Phi(\mathbf{x},t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(x')\delta(t-\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}dx'dy'dz'$$

$$\Phi(\mathbf{x},t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(t-\sqrt{x^2+(y-y')^2+(z-z')^2}/c)}{\sqrt{x^2+(y-y')^2+(z-z')^2}}dy'dz'$$

Because of the symmetry, we can set the origin anywhere we want on the sheet and we will still get the exact same answer. Set the origin aligned with the observation point so that $y = 0$, $z = 0$.

$$\Phi(\mathbf{x},t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(t-\sqrt{x^2+y'^2+z'^2}/c)}{\sqrt{x^2+y'^2+z'^2}}dy'dz'$$

Switch to polar coordinates in the y'-z' plane:

$$\Phi(\mathbf{x},t)=2\pi\int_0^\infty \frac{\delta(t-\sqrt{x^2+\rho'^2}/c)}{\sqrt{x^2+\rho'^2}}\rho' d\rho'$$

Again, we must work with the argument of the Dirac delta.

Set $x = \rho'$ and $f(\rho')=t-\sqrt{x^2+\rho'^2}/c$ so that $\frac{df}{d\rho'}=-\frac{1}{c}\frac{\rho'}{\sqrt{x^2+\rho'^2}}$.

The zeroes of this function are $\pm\sqrt{c^2t^2-x^2}$ and are only valid for $ct>|x|$

Putting all this together, we have:

$$\delta(t-\sqrt{x^2+\rho'^2}/c)=c^2t\frac{\delta(\rho'-\sqrt{c^2t^2-x^2})}{\sqrt{c^2t^2-x^2}}+c^2t\frac{\delta(\rho'+\sqrt{c^2t^2-x^2})}{\sqrt{c^2t^2-x^2}}$$

The radial coordinate can only positive and because the second Dirac delta exists at a negative radial point, it is zero in the entire domain, so that we have:

$$\delta(t-\sqrt{x^2+\rho'^2}/c)=c^2t\frac{\delta(\rho'-\sqrt{c^2t^2-x^2})}{\sqrt{c^2t^2-x^2}}$$

Using this in the integral, we have:

$\Phi(\mathbf{x},t)=2\pi c$ for $ct>|x|$ and zero otherwise.

or

$$\boxed{\Phi(x,t)=2\pi c\,\Theta(ct-|x|)}$$

This represents a plane traveling in the positive *x* direction and a plane traveling in the negative *x* direction, both starting at *x* = 0. Beyond these planes, the field is zero. At the planes, the observation points are getting their first signal due to the flash of sheet charge. Between the planes, the observation points are receiving signals from lateral locations far away on the sheet charge. Because they are far away, their signal took longer to reach the observation point. Interestingly, even though each lateral source point is farther away and its signal is therefore weaker, there are an ever increasing number of lateral source points to compensate. As a result, the wake of the causality shell is constant. This is an artifact of an infinite sheet charge. In reality, a finite sheet charge means that eventually there are no more lateral points left to flash and the field will die down at a fixed point.