Jackson 5.3 Homework Problem Solution

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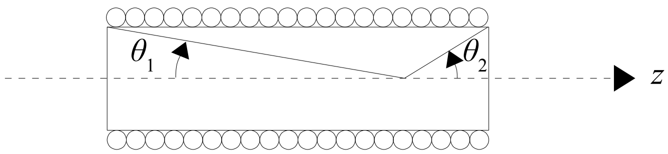
PROBLEM:

A right-circular solenoid of finite length L and radius a has N turns per unit length and carries a current I. Show that the magnetic induction on the cylinder axis in the limit $NL \to \infty$ is

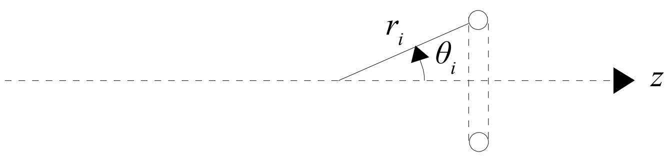
$$B_z = \frac{\mu_0 N I}{2}(\cos\theta_1 + \cos\theta_2)$$

where the angles are defined in the figure.



SOLUTION:

Let us first find the on-axis magnetic induction produced by one circular loop of current. Place the observation point such that the line from the observation point to the loop makes an angle θ_i with the solenoid's axis and is a distance r_i away. In this set-up, then the observation point can be considered at the origin of a spherical coordinate system.



If the current is flowing into the paper at the bottom and out of the paper at the top, the current density in spherical coordinates becomes:

$$\mathbf{J} = I\frac{\delta(r - r_i)}{r}\delta(\theta - \theta_i)\hat{\boldsymbol{\phi}}$$

Plug this into the Biot-Savart Law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3\mathbf{x}'$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\left[I\frac{\delta(r' - r_i)}{r'}\delta(\theta' - \theta_i)\hat{\boldsymbol{\phi}}'\right] \times (\mathbf{x} - \mathbf{x}')}{(r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')))^{3/2}} r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'$$

Spherical unit vectors are not constant or fixed, but change as we integrate. The safest thing to do is express the vector directions in Cartesian coordinates, using relations such as $\hat{\boldsymbol{\phi}}'=-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}}$ and

$$(\mathbf{x}-\mathbf{x}')=(r\sin\theta\cos\phi-r'\sin\theta'\cos\phi')\hat{\mathbf{i}}+(r\sin\theta\sin\phi-r'\sin\theta'\sin\phi')\hat{\mathbf{j}}+(r\cos\theta-r'\cos\theta')\hat{\mathbf{k}}$$

$$\mathbf{B}(\mathbf{x})=\frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r'^2\sin\theta' d r' d\theta' d\phi'\left[I\frac{\delta(r'-r_i)}{r'}\delta(\theta'-\theta_i)(-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}})\right]\times$$
$$\frac{((r\sin\theta\cos\phi-r'\sin\theta'\cos\phi')\hat{\mathbf{i}}+(r\sin\theta\sin\phi-r'\sin\theta'\sin\phi')\hat{\mathbf{j}}+(r\cos\theta-r'\cos\theta')\hat{\mathbf{k}})}{(r^2+r'^2-2rr'(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')))^{3/2}}$$

Now evaluate the Dirac delta's

$$\mathbf{B}(\mathbf{x})=\frac{I\mu_0}{4\pi}\int_0^{2\pi} r_i\sin\theta_i d\phi'[(-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}})]\times$$
$$\frac{((r\sin\theta\cos\phi-r_i\sin\theta_i\cos\phi')\hat{\mathbf{i}}+(r\sin\theta\sin\phi-r_i\sin\theta_i\sin\phi')\hat{\mathbf{j}}+(r\cos\theta-r_i\cos\theta_i)\hat{\mathbf{k}})}{(r^2+r_i^2-2rr_i(\cos\theta\cos\theta_i+\sin\theta\sin\theta_i\cos(\phi-\phi')))^{3/2}}$$

Evaluate the cross product

$$\mathbf{B}(\mathbf{x})=\frac{I\mu_0}{4\pi}\int_0^{2\pi} r_i\sin\theta_i d\phi'\frac{((r\cos\theta-r_i\cos\theta_i)(\cos\phi'\hat{\mathbf{i}}+\sin\phi'\hat{\mathbf{j}})+(r_i\sin\theta_i-r\sin\theta\cos(\phi-\phi'))\hat{\mathbf{k}})}{(r^2+r_i^2-2rr_i(\cos\theta\cos\theta_i+\sin\theta\sin\theta_i\cos(\phi-\phi')))^{3/2}}$$

This expression is general for any point in space in spherical coordinates. All that is left to do is perform the integral over the azimuthal angle and we would have the final solution to the total magnetic field (magnetic induction) **B** as any point in space created by a loop of current. The integral is messy however, and we don't need to do it for this particular problem. For an observation point at the origin $r=0$, this reduces to:

$$\mathbf{B}(\mathbf{x})=\frac{I\mu_0}{4\pi}\frac{\sin\theta_i}{r_i}\left[(-\cos\theta_i)(\hat{\mathbf{i}}\int_0^{2\pi}\cos\phi' d\phi'+\hat{\mathbf{j}}\int_0^{2\pi}\sin\phi' d\phi')+\sin\theta_i\hat{\mathbf{k}}\int_0^{2\pi}d\phi'\right]$$

We can now perform the integrals and find

$$\mathbf{B}(\mathbf{x})=\frac{\mu_0 I}{2}\frac{\sin^2\theta_i}{r_i}\hat{\mathbf{k}}$$

If we recognize that the ring has some fixed radius no matter where we place the ring, we can simplify this using $r_i\sin\theta_i=a$

$$\mathbf{B}(\mathbf{x})=\hat{\mathbf{k}}\frac{\mu_0 I a^2}{2}\frac{1}{r_i^3}$$

or

$$\mathbf{B}(\mathbf{x}) = \hat{\mathbf{k}} \frac{\mu_0 I a^2}{2} \frac{1}{(a^2+z_i^2)^{3/2}}$$

Now we can use the principle of superposition. If we have many of these wire loops very close together so that they fill the length $\Delta z = z_{i+1} - z_i$, but are in a small bundle still at this same location, and are spaced in a linear density N, then there are total $N\Delta z$ loops in this bundle. We simply multiply the equation above by to take into account that they are all at the same location but that there is no loops instead of one.

$$\mathbf{B}(\mathbf{x}) = \hat{\mathbf{k}} \frac{\mu_0 I a^2}{2} \frac{1}{(a^2+z_i^2)^{3/2}} N(z_{i+1} - z_i)$$

Using the principle of superposition again, we can add together all the magnetic fields due to many such bundles along the length of the solenoid to get the total field.

$$\mathbf{B}(\mathbf{x}) = \sum_{i=1}^{NL} \hat{\mathbf{k}} \frac{\mu_0 I a^2}{2} \frac{1}{(a^2+z_i^2)^{3/2}} N(z_{i+1} - z_i)$$

This is as far as we can get for a finite number of loops. However, if the number of loops is large, we can approximate the number of loops as approaching infinity, $NL \to \infty$. We can now shrink each bundle of width $(z_{i+1} - z_i)$ as small as we want. By definition, the sum becomes an integral and $\Delta z \to dz$

$$\mathbf{B}(\mathbf{x}) = \int_{z_1}^{z_2} \hat{\mathbf{k}} \frac{\mu_0 I a^2}{2} \frac{1}{(a^2+z^2)^{3/2}} N\, dz$$

$$\mathbf{B}(\mathbf{x}) = \hat{\mathbf{k}} \frac{\mu_0 N I a^2}{2} \int_{z_1}^{z_2} \frac{1}{(a^2+z^2)^{3/2}} dz$$

$$\mathbf{B}(\mathbf{x}) = \hat{\mathbf{k}} \frac{\mu_0 N I a^2}{2} \left[\frac{z}{a^2\sqrt{a^2+z^2}} \right]_{z_1}^{z_2}$$

$$\mathbf{B}(\mathbf{x}) = \hat{\mathbf{k}} \frac{\mu_0 N I a^2}{2} \left[\frac{z_2}{a^2\sqrt{a^2+z_2^2}} - \frac{z_1}{a^2\sqrt{a^2+z_1^2}} \right]$$

$$\mathbf{B}(\mathbf{x}) = \hat{\mathbf{k}} \frac{\mu_0 N I}{2} \left[\cos\theta_2 - \cos\theta_1^* \right]$$

The star on the angle 1 is to remind us that because we have used a fixed coordinate system, both angles are measured relative to the positive z-axis. But the problem asks us to define angle 1 as measured from the negative z-axis. We must make a transformation to get the angle defined in the way they want: $\theta_1^* \to \pi - \theta_1$

$$\boxed{\mathbf{B}(\mathbf{x}) = \hat{\mathbf{k}}\,\frac{\mu_0 N I}{2}\left[\cos\theta_2 + \cos\theta_1\right]}$$

We can check our answer by taking limiting cases. If the solenoid's length becomes infinite, both angles become zero and we get our usual answer for a long solenoid $\mathbf{B}(\mathbf{x}) = \hat{\mathbf{k}}\,\mu_0 N I$