Jackson 5.21 Homework Problem Solution

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PROBLEM:

A magnetostatic field is due entirely to a localized distribution of permanent magnetization.

(a) Show that $\int \mathbf{B}\cdot\mathbf{H}\, d^3x = 0$ provided the integral is taken over all space.

(b) From the potential energy (5.72) of a dipole in an external field, show that for a continuous distribution of permanent magnetization the magnetostatic energy can be written

$$W = \frac{\mu_0}{2}\int \mathbf{H}\cdot\mathbf{H}\, d^3x = -\frac{\mu_0}{2}\int \mathbf{M}\cdot\mathbf{H}\, d^3x$$

apart from an additive constant, which is independent of the orientation or position of the various constituent magnetized bodies.

SOLUTION:

(a) The keys concepts here are that there are no free currents anywhere, **J** = 0, so the **H** field must be irrotational:

$$\nabla\times\mathbf{H} = \mathbf{J} = 0$$

Also, because the magnetization is localized, all the fields must die down to zero at infinity.
Start by expanding the **B** field into the curl of the vector potential:

$$\int \mathbf{B}\cdot\mathbf{H}\, d^3x = \int \mathbf{H}\cdot(\nabla\times\mathbf{A})\, d^3x$$

Use a vector identity to shift around the curl operator:

$$\int \mathbf{B}\cdot\mathbf{H}\, d^3x = \int \nabla\cdot(\mathbf{A}\times\mathbf{H})\, d^3x + \int \mathbf{A}\cdot(\nabla\times\mathbf{H})\, d^3x$$

The curl of the H field is zero with no free currents so that the second integral on the right goes away.

$$\int \mathbf{B}\cdot\mathbf{H}\, d^3x = \int \nabla\cdot(\mathbf{A}\times\mathbf{H})\, d^3x$$

Use the divergence theorem to find:

$$\int \mathbf{B}\cdot\mathbf{H}\, d^3x = \int_{S \text{ at } \infty} \mathbf{n}\cdot(\mathbf{A}\times\mathbf{H})\, da$$

Because the volume integral is over all space, the surface containing it is at infinity. But because the sources are localized, the fields at infinity are zero, leading to:

$$\boxed{\int \mathbf{B}\cdot\mathbf{H}\, d^3 x = 0}$$

(b) The potential energy U of a dipole **m** in an external field **B** is:

$$W = -\mathbf{m}\cdot\mathbf{B}$$

If we take one dipole and then bring another one in from infinity, **B** will be caused by the first dipole. Assembling a whole collection of dipoles involves summing over all the potential energy of every dipole due to the field of every other dipole, divided by 2 to account for double-counting:

$$W = -\frac{1}{2}\sum_{i\neq j} \mathbf{m}_i\cdot\mathbf{B}_j$$

A continuous distribution of magnetization can be thought of as a collection of magnetic dipoles in the limit that they become very small and so close together that they are essentially continuous. At that point, the sum becomes an integral:

$$W = -\frac{1}{2}\int d\,\mathbf{m}\cdot\mathbf{B}$$

$$W = -\frac{1}{2}\int \left(\frac{d\,\mathbf{m}}{d^3\mathbf{x}}\right)\cdot\mathbf{B}\, d^3\mathbf{x}$$

$$W = -\frac{1}{2}\int \mathbf{M}\cdot\mathbf{B}\, d^3\mathbf{x}$$

Expand **B** according to $\mathbf{B} = \mu_0\mathbf{M} + \mu_0\mathbf{H}$:

$$W = -\frac{\mu_0}{2}\int \mathbf{M}\cdot\mathbf{M}\, d^3\mathbf{x} - \frac{\mu_0}{2}\int \mathbf{M}\cdot\mathbf{H}\, d^3\mathbf{x}$$

The first integral depends solely on the magnetization, which depends solely on the object. It can be thought of as an additive constant, which is independent of the orientation or position of the various constituent magnetized bodies, and can be ignored.

$$\boxed{W = -\frac{\mu_0}{2}\int \mathbf{M}\cdot\mathbf{H}\, d^3\mathbf{x}}$$

Expand **M** according to $\mathbf{M} = \mathbf{B}/\mu_0 - \mathbf{H}$:

$$W = -\frac{1}{2}\int \mathbf{B}\cdot\mathbf{H}\, d^3\mathbf{x} + \frac{\mu_0}{2}\int \mathbf{H}\cdot\mathbf{H}\, d^3\mathbf{x}$$

We showed in the previous section that $\int \mathbf{B}\cdot\mathbf{H}\, d^3 x = 0$ leading to:

$$\boxed{W = \frac{\mu_0}{2}\int \mathbf{H}\cdot\mathbf{H}\, d^3\mathbf{x}}$$

This equation is only valid if there are no free currents anywhere.