# Jackson 5.20 Homework Problem Solution

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**PROBLEM:**

(a) Starting from the force equation (5.12) and the fact that a magnetization **M** inside a volume $V$ bounded by a surface $S$ is equivalent to a volume current density $\mathbf{J}_M = (\nabla \times \mathbf{M})$ and a surface current density $(\mathbf{M} \times \mathbf{n})$, show that in the absence of macroscopic conduction currents the total magnetic force on a body can be written

$$\mathbf{F} = -\int_V (\nabla \cdot \mathbf{M}) \mathbf{B}_e \, d^3x + \int_S (\mathbf{M} \cdot \mathbf{n}) \mathbf{B}_e \, da$$

where $\mathbf{B}_e$ is the applied magnetic induction (not including that of the body in question). The force is now expressed in terms of the effective charge densities $\rho_M$ and $\sigma_M$. If the distribution of magnetization is not discontinuous, the surface can be at infinity and the force given by just the volume integral.

(b) A sphere of radius $R$ with uniform magnetization has its center at the origin of coordinates and its direction of magnetization making spherical angles $\theta_0$, $\phi_0$ . If the external magnetic field is the same as in Problem 5.11, use the expression of part a to evaluate the components of the force acting on the sphere.

**SOLUTION:**

(a) Starting with the general force expression:

$$\mathbf{F} = \int \mathbf{J}_{\text{tot}} \times \mathbf{B} \, d^3x$$

In the absence of macroscopic conduction currents (free currents), the only current present is the magnetization current $\mathbf{J}_M$. In this case, there is only an external field $\mathbf{B}_e$ and an object with a material response and/or original permanent magnetization, both described by the magnetization current.

$$\mathbf{F} = \int \mathbf{J}_M \times \mathbf{B}_e \, d^3x$$

Consider that the object occupies some finite volume $V$, so that outside $V$, the material response, and therefore the magnetization and magnetization current, are zero. Also, the surface $S$ that bounds the object of volume $V$ may have a discontinuity where we jump from object to not object. That gives rise to the possibility of a surface magnetization current $\mathbf{K}_M$ that can explicitly broken away from the total integral.

$$\mathbf{F} = \int_V \mathbf{J}_M \times \mathbf{B}_e \, d^3x + \oint_S \mathbf{K}_M \times \mathbf{B}_e \, da$$

Now plug in the representation of the currents in terms of magnetization, and switch the order of the cross products:

$$\mathbf{F}=-\int_V \mathbf{B}_e\times(\nabla\times\mathbf{M})\,d^3x-\oint_S \mathbf{B}_e\times(\mathbf{M}\times\mathbf{n})\,da$$

Recall the vector identity:

$$\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})$$

Arrange and label in an intuitive way to find:

$$\mathbf{B}_e\times(\nabla\times\mathbf{M})=\nabla(\mathbf{M}\cdot\mathbf{B}_e)-(\mathbf{M}\cdot\nabla)\mathbf{B}_e-(\mathbf{B}_e\cdot\nabla)\mathbf{M}-\mathbf{M}\times(\nabla\times\mathbf{B}_e)$$

In view of the relation $\nabla\times\mathbf{B}_e=\mu_0\mathbf{J}_e$ and the fact that the external currents $\mathbf{J}_e$ producing the external field are by definition not found in the volume of integration, we have $\nabla\times\mathbf{B}_e=0$ leading to:

$$\mathbf{B}_e\times(\nabla\times\mathbf{M})=\nabla(\mathbf{M}\cdot\mathbf{B}_e)-(\mathbf{M}\cdot\nabla)\mathbf{B}_e-(\mathbf{B}_e\cdot\nabla)\mathbf{M}$$

Plugging this in the force equation we have:

$$\mathbf{F}=-\int_V \nabla(\mathbf{M}\cdot\mathbf{B}_e)\,d^3x+\int_V(\mathbf{M}\cdot\nabla)\mathbf{B}_e\,d^3x+\int_V(\mathbf{B}_e\cdot\nabla)\mathbf{M}\,d^3x-\oint_S \mathbf{B}_e\times(\mathbf{M}\times\mathbf{n})\,da$$

Also use an identity on the second integral: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ which becomes

$$\mathbf{B}_e\times(\mathbf{M}\times\mathbf{n})=(\mathbf{B}_e\cdot\mathbf{n})\mathbf{M}-(\mathbf{B}_e\cdot\mathbf{M})\mathbf{n}$$

so that we have:

$$\mathbf{F}=-\int_V \nabla(\mathbf{M}\cdot\mathbf{B}_e)\,d^3x+\int_V(\mathbf{M}\cdot\nabla)\mathbf{B}_e\,d^3x+\int_V(\mathbf{B}_e\cdot\nabla)\mathbf{M}\,d^3x$$
$$-\oint_S(\mathbf{B}_e\cdot\mathbf{n})\mathbf{M}\,da+\oint_S(\mathbf{B}_e\cdot\mathbf{M})\mathbf{n}\,da$$

Use the divergence theorem on the first integral to get it into a surface integral and we find it cancels the last term, leading to:

$$\mathbf{F}=\int_V(\mathbf{M}\cdot\nabla)\mathbf{B}_e\,d^3x+\int_V(\mathbf{B}_e\cdot\nabla)\mathbf{M}\,d^3x-\oint_S(\mathbf{B}_e\cdot\mathbf{n})\mathbf{M}\,da$$

Using an integration by parts, you can show generally that:

$$\int_V(\mathbf{a}\cdot\nabla)\mathbf{b}\,dV=-\int_V(\nabla\cdot\mathbf{a})\mathbf{b}\,dV+\int_S(\hat{\mathbf{n}}\cdot\mathbf{a})\mathbf{b}\,da$$

Apply this to both volume integrals to find:

$$\mathbf{F}=-\int_V(\nabla\cdot\mathbf{M})\mathbf{B}_e\,dV+\oint_S(\hat{\mathbf{n}}\cdot\mathbf{M})\mathbf{B}_e\,da-\int_V(\nabla\cdot\mathbf{B}_e)\mathbf{M}\,dV$$

In general $\nabla\cdot\mathbf{B}_e=0$ leading to:

$$\boxed{\mathbf{F}=-\int_V(\nabla\cdot\mathbf{M})\mathbf{B}_e\,dV+\oint_S(\mathbf{M}\cdot\mathbf{n})\mathbf{B}_e\,da}$$

$$\boxed{\mathbf{F}=\int_V \rho_M \mathbf{B}_e\, dV+\oint_S \sigma_M \mathbf{B}_e\, da \text{ where } \rho_M=-\nabla\cdot\mathbf{M} \text{ and } \sigma_M=\mathbf{M}\cdot\mathbf{n}}$$

We can therefore treat a magnetized object as a collection of pseudo-magnetic charges and have these charges experience forces in the usual way.

(b) First of all, the magnetization is uniform inside the sphere, so there is no effective magnetic volume charge density:

$$\rho_M=0$$

There is an effective magnetic surface charge density on the sphere because of the discontinuity of magnetization at the surface. Let as find the surface charge density:

$$\sigma_M=\mathbf{M}\cdot\mathbf{n}$$

$$\sigma_M=M_0\hat{\mathbf{r}}_0\cdot\hat{\mathbf{r}}$$

$$\sigma_M=M_0(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos\phi\cos\phi_0+\sin\theta\sin\theta_0\sin\phi\sin\phi_0)$$

Define $x_0=\sin\theta_0\cos\phi_0,\ y_0=\sin\theta_0\sin\phi_0,\ z_0=\cos\theta_0$ so that:

$$\sigma_M=M_0(z_0\cos\theta+x_0\sin\theta\cos\phi+y_0\sin\theta\sin\phi)$$

The applied magnetic field is the one from 5.11:

$$\mathbf{B}=B_0\left[(1+\beta y)\hat{\mathbf{i}}+(1+\beta x)\hat{\mathbf{j}}\right]$$

$$\mathbf{B}=B_0\left[(\hat{\mathbf{i}}+\hat{\mathbf{j}})+\beta r\sin\theta(\hat{\mathbf{i}}\sin\phi+\hat{\mathbf{j}}\cos\phi)\right]$$

Plug all of these in the force equation found in part a:

$$\mathbf{F}=\int_V \rho_M \mathbf{B}_e\, dV+\oint_S \sigma_M \mathbf{B}_e\, da$$

$$\mathbf{F}=\oint_S \sigma_M \mathbf{B}_e\, da$$

$$\mathbf{F}=\int_0^{2\pi}\int_0^{\pi} M_0(z_0\cos\theta+x_0\sin\theta\cos\phi+y_0\sin\theta\sin\phi)B_0\left[(\hat{\mathbf{i}}+\hat{\mathbf{j}})+\beta R\sin\theta(\hat{\mathbf{i}}\sin\phi+\hat{\mathbf{j}}\cos\phi)\right]R^2\sin\theta\, d\theta\, d\phi$$

After distributing out all factors, and doing many integrals, we find:

$$\mathbf{F}=M_0B_0\beta\frac{4}{3}\pi R^3(y_0\hat{\mathbf{i}}+x_0\hat{\mathbf{j}}) \quad \text{or} \quad \boxed{\mathbf{F}=M_0B_0\beta V(\sin\theta_0\sin\phi_0\hat{\mathbf{i}}+\sin\theta_0\cos\phi_0\hat{\mathbf{j}})}$$