Jackson 5.1 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell

PROBLEM:

Starting with the differential expression

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

for the magnetic induction at the point P with coordinate $\mathbf{x}$ produced by an increment of current $I\, d\mathbf{l}'$ at $\mathbf{x}'$, show explicitly that for a closed loop carrying a current I the magnetic induction at P is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla \Omega$$

where Ω is the solid angle subtended by the loop at the point P. This corresponds to a magnetic scalar potential, $\Phi_M = -\mu_0 I\Omega/4\pi$. The sign convention for the solid angle Ω is positive if the point P views the "inner" side of the surface spanning the loop, that is, if a unit normal $\mathbf{n}$ to the surface is defined by the direction of current flow via the right-hand rule, Ω is positive if n points away from the point P, and negative otherwise. This is the same convention as in Section 1.6 for the electric dipole layer.

SOLUTION:

Start with the differential expression and put all the constants on the other side to get them out of the way:

$$\frac{4\pi}{\mu_0 I} d\mathbf{B} = d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

This is a vector equation that must hold for all components. If we take one component in a general way, then it will apply to all components. Let us take the i^{th} Cartesian component.

$$\frac{4\pi}{\mu_0 I} dB_i = \hat{\mathbf{x}}_i \cdot \left[d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \right]$$

now integrate over a closed loop to get the total field due to the loop:

$$\frac{4\pi}{\mu_0 I} B_i = \oint \hat{\mathbf{x}}_i \cdot \left[d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \right]$$

We want to try to do this integral. Use the identity $\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla' \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right)$

$$\frac{4\pi}{\mu_0 I}B_i=\oint \hat{\mathbf{x}}_i\cdot\left[d\mathbf{l}'\times\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]$$

Use the vector identity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

$$\frac{4\pi}{\mu_0 I}B_i=\oint d\mathbf{l}'\cdot\left[\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{\mathbf{x}}_i\right]$$

Use Stoke's theorem to convert the line integral to an area integral over the surface bounded by the closed line integral.

$$\frac{4\pi}{\mu_0 I}B_i=\int\left[\nabla'\times\left[\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{\mathbf{x}}_i\right]\right]\cdot\hat{\mathbf{n}}'\,da'$$

Now use the identity:

$$\nabla'\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}(\nabla'\cdot\mathbf{b})-\mathbf{b}(\nabla'\cdot\mathbf{a})+(\mathbf{b}\cdot\nabla')\mathbf{a}-(\mathbf{a}\cdot\nabla')\mathbf{b}$$

and set $\mathbf{a}=\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)$ and $\mathbf{b}=\hat{\mathbf{x}}_i$

$$\nabla'\times(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\hat{\mathbf{x}}_i)=-\hat{\mathbf{x}}_i(\nabla'^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right))+(\hat{\mathbf{x}}_i\cdot\nabla')\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)$$

Use this identity:

$$\frac{4\pi}{\mu_0 I}B_i=-\int\left[\hat{\mathbf{x}}_i(\nabla'^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right))\right]\cdot\hat{\mathbf{n}}'\,da'+\int\left[(\hat{\mathbf{x}}_i\cdot\nabla')\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]\cdot\hat{\mathbf{n}}'\,da'$$

The Laplacian of $1/R$ is related to the Dirac delta:

$$\frac{4\pi}{\mu_0 I}B_i=-\int[\hat{\mathbf{x}}_i(-4\pi\delta(\mathbf{x}-\mathbf{x}'))]\cdot\hat{\mathbf{n}}'\,da'+\int\left[(\hat{\mathbf{x}}_i\cdot\nabla')\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]\cdot\hat{\mathbf{n}}'\,da'$$

Assume the observation point $\mathbf{x}$ never lies in the current loop's area of integration. Then the Dirac delta is always zero and the first term goes away.

$$\frac{4\pi}{\mu_0 I}B_i=\int\left[\frac{\partial}{\partial x_i'}\left[\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]\right]\cdot\hat{\mathbf{n}}'\,da'$$

Due to the symmetry of the factor contained in the inner brackets, we can make the partial derivative with respect to unprimed coordinates if we negate the entire expression. We can then bring the partial derivative out of the integral:

$$\frac{4\pi}{\mu_0 I}B_i = -\frac{\partial}{\partial x_i}\int\left[\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]\cdot\hat{\mathbf{n}}'\,da'$$

Now again use $\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)$

$$\frac{4\pi}{\mu_0 I}B_i = -\frac{\partial}{\partial x_i}\int\left[\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right]\cdot\hat{\mathbf{n}}'\,da'$$

$$\frac{4\pi}{\mu_0 I}B_i = \frac{\partial}{\partial x_i}\int\left[\frac{\cos\gamma}{|\mathbf{x}-\mathbf{x}'|^2}\right]da'$$

Here the angle is between the normal vector to the surface patch and the separation vector connecting the observation point and the patch. A quick sketch of the term in brackets reveals that it is the solid angle subtended by the patch in relation to the observation point.

$$\frac{4\pi}{\mu_0 I}B_i = \frac{\partial}{\partial x_i}\int d\Omega'$$

$$\frac{4\pi}{\mu_0 I}B_i = \frac{\partial}{\partial x_i}\Omega(\mathbf{x})$$

Combing all components, we get:

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi}\nabla\Omega}$$