Jackson 5.19 Homework Problem Solution

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PROBLEM:

A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0, uniform through-out its volume and parallel to its axis.

(a) Determine the magnetic field **H** and magnetic induction **B** at all points on the axis of the cylinder, both inside and outside.

(b) Plot the ratios $\mathbf{B}/\mu_0 M_0$ and $\mathbf{H}/M_0$ on the axis as functions of z for $L/a = 5$

SOLUTION:

(a) Let us place the center of the cylinder at the origin and its axis aligned with the z axis. Then it is natural to use cylindrical coordinates. Let us state everything we know about this problem outright. There are no free currents present, so there cannot be any curl to the **H** field.

$$\nabla \times \mathbf{H} = 0$$

This is valid inside and out. The curl of the gradient of any function is always zero, so we can define a magnetic scalar potential

$$\mathbf{H} = -\nabla \Phi_M$$

Outside the object, there is no material, so that the magnetization outside must be zero and the total field **B** depends only on the **H** field.

$$\mathbf{B} = \mu_0 \mathbf{H} \quad \text{and } \mathbf{M} = 0 \text{ outside}$$

We should also note that there are no applied external fields, only the fields created by the shape. This means that all fields should die down to zero infinitely away from the shape.
Inside, the magnetization is a known:

$$\mathbf{M} = M_0 \hat{\mathbf{z}} \quad \text{inside}$$

Generally speaking, the equation that says there are no magnetic monopoles can be cast as a relationship between **M** and **H:**

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \nabla \cdot (\mu_0 \mathbf{H} + \mu_0 \mathbf{M}) = 0 \quad \rightarrow \quad \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

This is a totally general results, applicable everywhere, no matter what material is present. It means that whatever divergence that may exist in the magnetization must be perfectly canceled by the divergence in the **H** field in order to keep the actual, real total field **B** divergence-less. We now substitute in the definition of **H** in terms of the magnetic scalar potential and recognize an entity that acts like a magnetic charge density:

$$\nabla^2 \Phi_M = -\rho_M \quad \text{where} \quad \rho_M = -\nabla \cdot \mathbf{M}$$

Because the magnetization is constant inside the cylinder, its divergence is zero there. The magnetic charge density must reside on the surface. Drawing a closed surface around the surface charge density and shrinking it down leads to $(\mathbf{M}_2 - \mathbf{M}_1)\cdot\mathbf{n} = -\sigma_M$. Outside there is no magnetization $\mathbf{M}_2 = 0$ and inside we know $\mathbf{M}_1 = M_0 \hat{\mathbf{z}}$ so that we have:

$$\sigma_M = M_0 \hat{\mathbf{z}} \cdot \mathbf{n}$$

$\sigma_M = M_0$ on the top of the cylinder, $\sigma_M = -M_0$ on the bottom of the cylinder, and $\sigma_M = 0$ on the sides

This is best summed up as:

$$\rho_M = M_0(\delta(z - L/2) - \delta(z + L/2)) \quad \text{for } \rho < a \text{, and } \rho_M = 0 \text{ for } \rho > a.$$

The Poisson equation for the magnetic scalar potential has the general solution:

$$\Phi_M = \frac{1}{4\pi} \int \frac{\rho_M}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

$$\Phi_M = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{\rho_M}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z - z')^2}} \rho' d\rho' d\phi' dz'$$

$$\Phi_M = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{a} \frac{(M_0(\delta(z' - L/2) - \delta(z' + L/2)))}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z - z')^2}} \rho' d\rho' d\phi' dz'$$

$$\Phi_M = \frac{M_0}{4\pi} \left[\int_0^{2\pi} \int_0^{a} \frac{\rho' d\rho' d\phi'}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z - L/2)^2}} - \int_0^{2\pi} \int_0^{a} \frac{\rho' d\rho' d\phi'}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi') + (z + L/2)^2}} \right]$$

The problem only asks for the solution on the axis of the cylinder, so we can safely set $\rho = 0$.

$$\Phi_M = \frac{M_0}{4\pi} \left[\int_0^{2\pi} \int_0^{a} \frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2 + (z - L/2)^2}} - \int_0^{2\pi} \int_0^{a} \frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2 + (z + L/2)^2}} \right]$$

$$\Phi_M = \frac{M_0}{2} \left[\int_0^{a} \frac{\rho' d\rho'}{\sqrt{\rho'^2 + (z - L/2)^2}} - \int_0^{a} \frac{\rho' d\rho'}{\sqrt{\rho'^2 + (z + L/2)^2}} \right]$$

$$\Phi_M = \frac{M_0}{2} \left[\left[\sqrt{\rho'^2 + (z - L/2)^2} \right]_0^a - \left[\sqrt{\rho'^2 + (z + L/2)^2} \right]_0^a \right]$$

$$\boxed{\Phi_M = \frac{M_0}{2} \left[\sqrt{a^2 + (z - L/2)^2} - \sqrt{a^2 + (z + L/2)^2} - |z - L/2| + |z + L/2| \right]}$$

$$\mathbf{H} = -\nabla\Phi_M$$

$$\mathbf{H} = -\frac{\partial\Phi_M}{\partial z}\hat{\mathbf{z}}$$

$$\mathbf{H} = -\frac{M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{|z-L/2|} + \frac{z+L/2}{|z+L/2|}\right]\hat{\mathbf{z}}$$

A careful evaluation of the absolute value in the different regions leads to:

$$\boxed{\mathbf{H}_{\text{out}} = -\frac{M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right]\hat{\mathbf{z}}} \quad \text{on axis}$$

$$\boxed{\mathbf{H}_{\text{in}} = -\frac{M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} + 2\right]\hat{\mathbf{z}}} \quad \text{on axis}$$

Using $\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}$, $\mathbf{M} = 0$ outside, and $\mathbf{M} = M_0\hat{\mathbf{z}}$ inside

$$\boxed{\mathbf{B} = -\frac{\mu_0 M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right]\hat{\mathbf{z}}} \quad \text{on axis}$$

The total field inside and outside end up having the same form

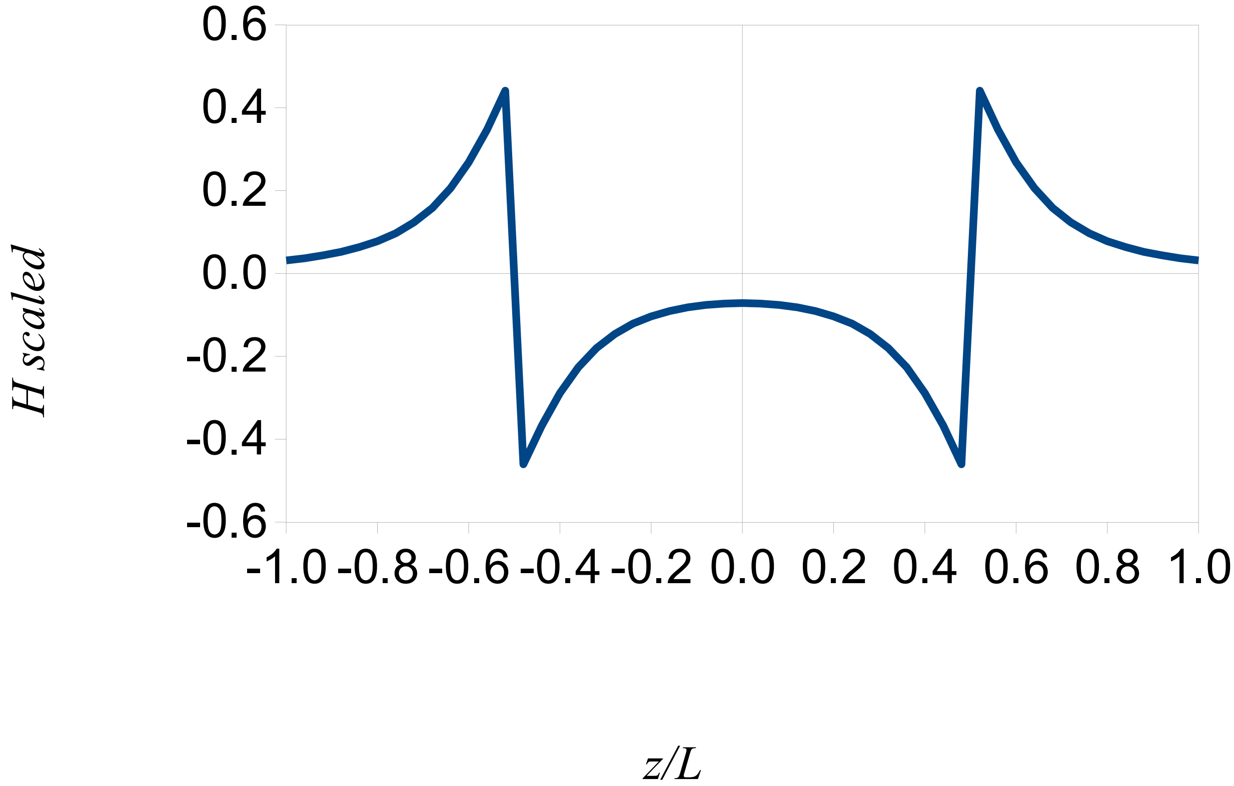
(b) Plot the ratios $\mathbf{B}/\mu_0 M_0$ and $\mathbf{H}/M_0$ on the axis as functions of z for $L/a = 5$

For $L/a = 5$ we have

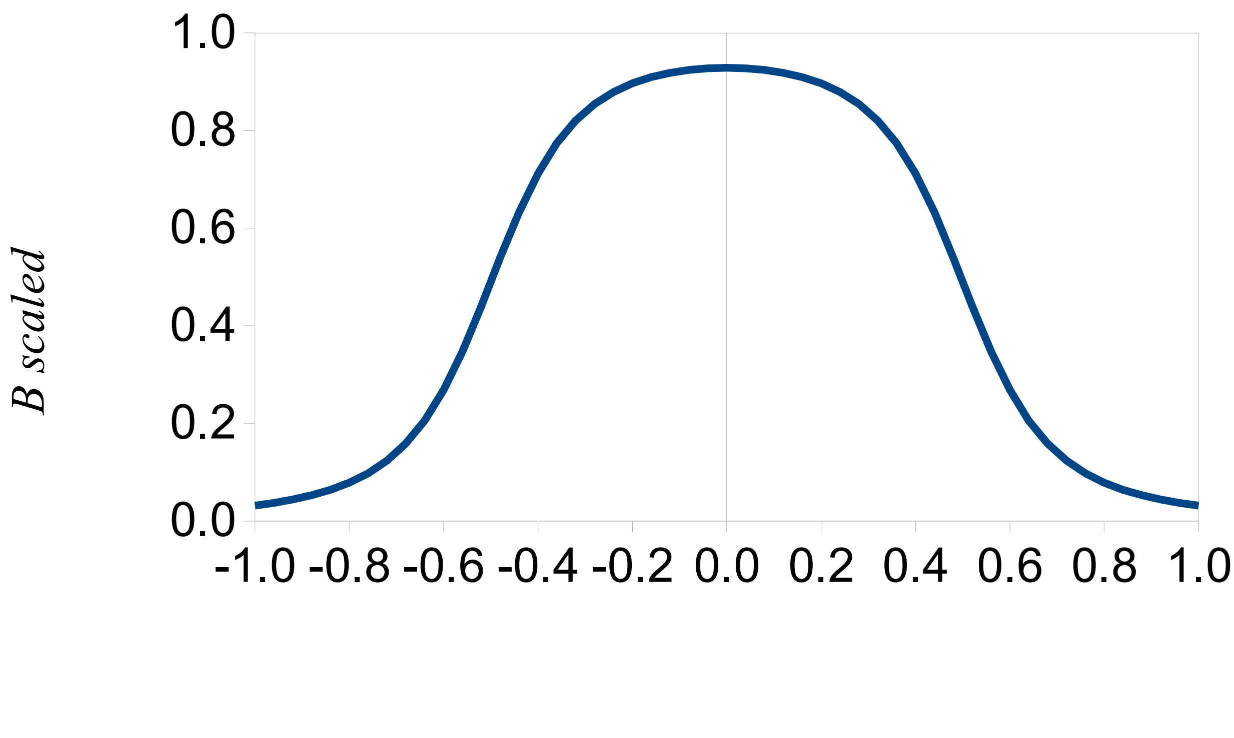
$$\mathbf{H}_{\text{out}}/M_0 = -\frac{1}{2}\left[\frac{z/L-1/2}{\sqrt{1/25+(z/L-1/2)^2}} - \frac{z/L+1/2}{\sqrt{1/25+(z/L+1/2)^2}}\right]\hat{\mathbf{z}} \quad \text{on axis}$$

$$\mathbf{H}_{\text{in}}/M_0 = -\frac{1}{2}\left[\frac{z/L-1/2}{\sqrt{1/25+(z/L-1/2)^2}} - \frac{z/L+1/2}{\sqrt{1/25+(z/L+1/2)^2}} + 2\right]\hat{\mathbf{z}} \quad \text{on axis}$$

$$\mathbf{B}/\mu_0 M_0 = -\frac{1}{2}\left[\frac{z/L-1/2}{\sqrt{1/25+(z/L-1/2)^2}} - \frac{z/L+1/2}{\sqrt{1/25+(z/L+1/2)^2}}\right]\hat{\mathbf{z}} \quad \text{on axis}$$



H scaled
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
z/L



B scaled
1.0
0.8
0.6
0.4
0.2
0.0
-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
z/L