Jackson 5.18 Homework Problem Solution

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PROBLEM:

A circular loop of wire having a radius a and carrying a current I is located in vacuum with its center a distance d away from a semi-infinite slab of permeability μ. Find the force acting on the loop when

(a) the plane of the loop is parallel to the face of the slab,

(b) the plane of the loop is perpendicular to the face of the slab

(c) Determine the limiting form of your answer to parts a and b when $d >> a$. Can you obtain these limiting values in some simple and direct way?

SOLUTION:

(a) We can place the surface of the semi-infinite slab along the $z = 0$ plane, and center the loop on the z-axis. We can split the problem into to two separate regions and find the solution in each region and then link them using boundary conditions. In the region where $z > 0$, there is no magnetic material, so we can use all the regular equations that do not take materials into effect. We use the method of images in this region to simulate the effects of the boundary. Place a circular loop of wire having a radius a and carrying a current I' centered at the location $z = -d$.

The vector potential due to a circular loop in vacuum located at $z = 0$ was found previously to be:

$$\mathbf{A} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{a^2 + r^2 - 2 a r \sin\theta \cos\phi'}}$$

Switch this into Cartesian coordinates:

$$\mathbf{A} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{a^2 + x^2 + y^2 + z^2 - 2 a \sqrt{x^2 + y^2} \cos\phi'}}$$

Now use a translated form of this for each ring and switch back to spherical coordinates:

$$\mathbf{A}_{\text{upper}} = \hat{\boldsymbol{\phi}} \frac{\mu_0 a}{4\pi} \int_0^{2\pi} d\phi' \cos\phi' \Big[I \frac{1}{\sqrt{a^2 + d^2 + r^2 - 2 r \cos\theta\, d - 2 a r \sin\theta \cos\phi'}} + I' \frac{1}{\sqrt{a^2 + d^2 + r^2 + 2 r \cos\theta\, d - 2 a r \sin\theta \cos\phi'}} \Big]$$

For the region $z < 0$, there is no current, so we only need a image current I'' at $z = d$ to simulate the effects of the boundary, leading to the vector potential:

$$\mathbf{A}_{\text{lower}} = \hat{\boldsymbol{\phi}} \frac{\mu a}{4\pi} \int_0^{2\pi} d\phi' \cos\phi' I'' \frac{1}{\sqrt{a^2+d^2+r^2-2r\cos\theta\, d-2ar\sin\theta\cos\phi'}}$$

Now match up both regions using the boundary conditions

$$\left[\mathbf{B}_{\text{upper}} \cdot \mathbf{n} = \mathbf{B}_{\text{lower}} \cdot \mathbf{n}\right]_{\text{on } S}$$

$$\left[(\nabla \times \mathbf{A}_{\text{upper}}) \cdot \hat{\boldsymbol{\theta}} = (\nabla \times \mathbf{A}_{\text{lower}}) \cdot \hat{\boldsymbol{\theta}}\right]_{\theta=\pi/2}$$

$$\left[\frac{\partial}{\partial r}(r A_{\phi,\text{ upper}}) = \frac{\partial}{\partial r}(r A_{\phi,\text{ lower}})\right]_{\theta=\pi/2}$$

After much algebra, this leads to:

$$\boxed{\mu_0(I+I') = \mu I''}$$

The other boundary condition is:

$$\left[\mathbf{n} \times \mathbf{H}_{\text{upper}} = \mathbf{n} \times \mathbf{H}_{\text{lower}}\right]_{\text{on } S}$$ (because there is no free current at the boundary surface)

$$\left[\frac{1}{\mu_0} \hat{\boldsymbol{\theta}} \times (\nabla \times \mathbf{A}_{\text{upper}}) = \frac{1}{\mu} \hat{\boldsymbol{\theta}} \times (\nabla \times \mathbf{A}_{\text{lower}})\right]_{\theta=\pi/2}$$

$$\left[\frac{1}{\mu_0} \frac{\partial}{\partial\theta}(\sin\theta A_{\phi,\text{upper}}) = \frac{1}{\mu} \frac{\partial}{\partial\theta}(\sin\theta A_{\phi,\text{ lower}})\right]_{\theta=\pi/2}$$

After much algebra, this leads to:

$$\boxed{I - I' = I''}$$

The two equations in boxes give us enough information to solve the problem uniquely:

$$\boxed{I' = \frac{\mu-\mu_0}{\mu+\mu_0} I}$$

$$\boxed{I'' = \frac{2\mu_0}{\mu+\mu_0} I}$$

The final solution in the upper region (which is all we need to calculate the force) for the vector potential is then:

$$\mathbf{A}_{\text{upper}} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} d\phi' \cos\phi' [\frac{1}{\sqrt{a^2 + d^2 + r^2 - 2r\cos\theta\, d - 2\, a r \sin\theta\cos\phi'}}$$
$$+ \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right) \frac{1}{\sqrt{a^2 + d^2 + r^2 + 2r\cos\theta\, d - 2\, a\, r\sin\theta\cos\phi'}}]$$

Let us to switch to cylindrical coordinates:

$$\mathbf{A}_{\text{upper}} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} d\phi' \cos\phi' [\frac{1}{\sqrt{a^2 + d^2 + \rho^2 + z^2 - 2\, z\, d - 2\, a\, \rho\cos\phi'}}$$
$$+ \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right) \frac{1}{\sqrt{a^2 + d^2 + \rho^2 + z^2 + 2\, z\, d - 2\, a\, \rho\cos\phi'}}]$$

The force on a wire due to a magnetic field is:

$$\mathbf{F} = I \oint d\,\mathbf{l} \times \mathbf{B}$$

In this case the magnetic field is constant around the ring, so this reduces to

$$\mathbf{F} = \left[2\pi\, a\, I\, \hat{\boldsymbol{\phi}} \times \mathbf{B}_{\text{upper}}\right]_{\text{on wire}}$$

$$\mathbf{F} = \left[2\pi\, a\, I\, \hat{\boldsymbol{\phi}} \times (\nabla \times \mathbf{A}_{\text{upper}})\right]_{\text{on wire}}$$

In cylindrical components this becomes:

$$\mathbf{F} = 2\pi\, a\, I \left[\hat{\mathbf{z}} \frac{\partial}{\partial z} A_\phi\right]_{\rho = a,\, z = d}$$ where all other components are dropped due to symmetry.

Upon evaluating the derivative we find.

$$\boxed{\mathbf{F} = -\hat{\mathbf{z}}\mu_0 I^2 a^2 d \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right) \int_0^{2\pi} \frac{\cos\phi'}{(4d^2 + 2a^2(1 - \cos\phi'))^{3/2}} d\phi'}$$

We should note that the integral of the first term equated to zero. This makes sense because the forces of the real wire on itself should all cancel out due to the symmetry.

(b) the plane of the loop is perpendicular to the face of the slab

This is very similar to the previous problem. We again use an image current, but must be careful of the geometry. Let us place the real loop in the *x*-*y* plane centered at the origin, and the material boundary at $x = d$. The image loop is then centered at $x = 2d$.
The potential due to the real loop is:

$$\mathbf{A}=\hat{\boldsymbol{\phi}}\frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos\phi'}}$$

In Cartesian coordinates this is:

$$\mathbf{A}=\frac{(-y\hat{\mathbf{i}}+x\hat{\mathbf{j}})}{\sqrt{x^2+y^2}}\frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{a^2+x^2+y^2+z^2-2a\sqrt{x^2+y^2}\cos\phi'}}$$

The image loop's potential **A'** is the same except with a current I' and shifted $x \to x-2d$

$$\mathbf{A}'=\frac{(-y\hat{\mathbf{i}}+(x-2d)\hat{\mathbf{j}})}{\sqrt{(x-2d)^2+y^2}}\frac{\mu_0 I' a}{4\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{a^2+(x-2d)^2+y^2+z^2-2a\sqrt{(x-2d)^2+y^2}\cos\phi'}}$$

Using the arguments in the first part, we can show that:

$$I'=\frac{\mu-\mu_0}{\mu+\mu_0}I$$

When calculating the force, the force of the wire on itself will cancel out due to the symmetry, so that we only need to keep the field due to the image current:

$$\mathbf{A}'=\frac{\mu-\mu_0}{\mu+\mu_0}I\frac{(-y\hat{\mathbf{i}}+(x-2d)\hat{\mathbf{j}})}{\sqrt{(x-2d)^2+y^2}}\frac{\mu_0 a}{4\pi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{a^2+(x-2d)^2+y^2+z^2-2a\sqrt{(x-2d)^2+y^2}\cos\phi'}}$$

The math will simplify best if we put everything in cylindrical coordinates:

$$\mathbf{A}'=\frac{\mu-\mu_0}{\mu+\mu_0}I\frac{((-2d\sin\phi)\hat{\boldsymbol{\rho}}+(\rho-2d\cos\phi)\hat{\boldsymbol{\phi}})}{\sqrt{\rho^2+4d^2-4d\rho\cos\phi}}\frac{\mu_0 a}{4\pi}$$
$$\int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{a^2+4d^2+\rho^2-4d\rho\cos\phi+z^2-2a\sqrt{\rho^2+4d^2-4d\rho\cos\phi}\cos\phi'}}$$

The force on a wire due to a magnetic field is:

$$\mathbf{F}=I\oint d\mathbf{l}\times\mathbf{B}$$

In this case, the magnetic field is not constant around the loop, so we can not remove it from the integral. We must do the integral outright. Expand in cylindrical coordinates:

$$\mathbf{F}=Ia\int_0^{2\pi} d\phi\,\hat{\boldsymbol{\phi}}\times\mathbf{B}$$

$$\mathbf{F}=Ia\int_0^{2\pi} d\phi(B_z\hat{\boldsymbol{\rho}}-B_\rho\hat{\mathbf{z}})$$

$$\mathbf{F}=I\,a\int_0^{2\pi} d\,\phi\left(\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi{}')-\frac{1}{\rho}\frac{\partial}{\partial\phi}A_\rho{}'\right)\hat{\boldsymbol{\rho}}-\left(\frac{-\partial}{\partial z}A_\phi{}'\right)\hat{\mathbf{z}}\right)$$

$$\mathbf{F}=I\,a\int_0^{2\pi} d\,\phi\left(\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi{}')-\frac{1}{\rho}\frac{\partial}{\partial\phi}A_\rho{}'\right)(\cos\phi\,\hat{\mathbf{i}}+\sin\phi\,\hat{\mathbf{j}})-\left(\frac{-\partial}{\partial z}A_\phi{}'\right)\hat{\mathbf{z}}\right)$$

From the symmetry of the problem, the only component that should remain is in the x direction:

$$\mathbf{F}=\hat{\mathbf{i}}\,I\,a\int_0^{2\pi} d\,\phi\cos\phi\left(\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi{}')-\frac{1}{\rho}\frac{\partial}{\partial\phi}A_\rho{}'\right)\right)\quad\text{where}$$

$$\mathbf{A}'=\frac{\mu-\mu_0}{\mu+\mu_0}I\frac{((-2d\sin\phi)\hat{\boldsymbol{\rho}}+(\rho-2d\cos\phi)\hat{\boldsymbol{\phi}})}{\sqrt{\rho^2+4d^2-4d\,\rho\cos\phi}}\frac{\mu_0 a}{4\pi}\int_0^{2\pi} d\,\phi'\frac{\cos\phi'}{\sqrt{a^2+4d^2+\rho^2-4d\,\rho\cos\phi+z^2-2a\sqrt{\rho^2+4d^2-4d\,\rho\cos\phi}\cos\phi'}}$$

We must remember that the loop integral is done at $\rho = a$ and $z = 0$. The derivatives have no dependence on z so we can safely set $z = 0$ before doing the derivatives.

$$\boxed{\mathbf{F}=\left[\hat{\mathbf{i}}\,I\int_0^{2\pi} d\,\phi\cos\phi\left(\left(\frac{\partial}{\partial\rho}(\rho A_\phi{}')-\frac{\partial}{\partial\phi}A_\rho{}'\right)\right)\right]_{\rho=a}}\quad\text{where}$$

$$\mathbf{A}'=\frac{\mu-\mu_0}{\mu+\mu_0}I\frac{\mu_0 a}{4\pi}\frac{((-2d\sin\phi)\hat{\boldsymbol{\rho}}+(\rho-2d\cos\phi)\hat{\boldsymbol{\phi}})}{\sqrt{\rho^2+4d^2-4d\,\rho\cos\phi}}\int_0^{2\pi} d\,\phi'\frac{\cos\phi'}{\sqrt{a^2+4d^2+\rho^2-4d\,\rho\cos\phi-2a\sqrt{\rho^2+4d^2-4d\,\rho\cos\phi}\cos\phi'}}$$

The integrals cannot be done, so it is not that helpful to go much further.

(c) Determine the limiting form of your answer to parts a and b when $d >> a$. Can you obtain these limiting values in some simple and direct way?

For the first case, we take the solution and get everything in terms of the ratio a/d so that we can use the statement $a/d << 1$.

$$\mathbf{F}=-\hat{\mathbf{z}}\mu_0 I^2(a/d)^2\frac{1}{8}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\int_0^{2\pi}\frac{\cos\phi'}{(1+1/2(a/d)^2(1-\cos\phi'))^{3/2}}d\,\phi'$$

Use the Taylor series expansion and because $a/d << 1$ we can keep only the first two terms:

$$(1+1/2(a/d)^2(1-\cos\phi'))^{-3/2}=1-\frac{3}{4}(a/d)^2(1-\cos\phi')$$

$$\mathbf{F}=-\hat{\mathbf{z}}\mu_0 I^2 (a/d)^2 \frac{1}{8}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\int_0^{2\pi}\cos\phi'(1-\frac{3}{4}(a/d)^2(1-\cos\phi'))d\phi'$$

$$\boxed{\mathbf{F}=-\hat{\mathbf{z}}\frac{3\pi}{32}\mu_0 I^2\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\left(\frac{a}{d}\right)^4}$$

For paramagnetic materials like steel, the permeability is greater than the permeability of free space, $\mu > \mu_0$ so that the force is in the negative z direction, or in other words, is attracted to the slab of paramagnetic material. Because of the I^2, this is true no matter which direction the current is flowing in. the loop of wire is a magnetic dipole and can be thought of as a little bar magnet being attracted to paramagnetic materials. For diamagnetic materials, $\mu < \mu_0$, the force is repulsive.

When $d >> a$, we can treat the current as a far-away localized current distribution. Then we could have directly used:

$\mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B}(0))$ to get the same answer.

For the second case, we use the same approach.

$$\mathbf{A}'=\frac{\mu-\mu_0}{\mu+\mu_0}I\frac{\mu_0 a/d}{4\pi}\frac{((-\sin\phi)\hat{\boldsymbol{\rho}}+((1/2)\rho/d-\cos\phi)\hat{\boldsymbol{\phi}})}{\sqrt{1-(\rho/d)\cos\phi}}\int_0^{2\pi}d\phi'\frac{\cos\phi'}{\sqrt{1-(\rho/d)\cos\phi-(a/d)\cos\phi'}}$$

Use the Taylor series expansion:

$$(1-x)^{-1/2}=1+\frac{1}{2}x+...$$

$$(1-((\rho/d)\cos\phi+(a/d)\cos\phi'))^{(-1/2)}=1+\frac{1}{2}((\rho/d)\cos\phi+(a/d)\cos\phi')$$

and $(1-(\rho/d)\cos\phi)^{-1/2}=1+\frac{1}{2}((\rho/d)\cos\phi)$

The higher order terms were dropped. After the expansions are used and the integral is evaluated:

$$\mathbf{A}'=\frac{\mu-\mu_0}{\mu+\mu_0}I\frac{\mu_0(a/d)^2}{8}((-\sin\phi)\hat{\boldsymbol{\rho}}+((1/2)\rho/d-\cos\phi)\hat{\boldsymbol{\phi}})(1+\frac{1}{2}((\rho/d)\cos\phi))$$

Plug this now into the rest of the equation:

$$\mathbf{F}=\left[\hat{\mathbf{i}}I\int_0^{2\pi}d\phi\cos\phi((\frac{\partial}{\partial\rho}(\rho A_\phi')-\frac{\partial}{\partial\phi}A_\rho'))\right]_{\rho=a}$$

After much algebra:

$$\boxed{\mathbf{F} = \hat{\mathbf{i}}\,\frac{3\pi}{32}\mu_0 I^2 \left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\left(\frac{a}{d}\right)^4}$$

This is the exact same result as for the first case when we realize that the slab was put on the positive x axis in this case and was put on the negative z axis in the first case.

We can make the general conclusion that a loop of current, and by extension, a magnetic dipole, will always be attracted to a slab of paramagnetic (and ferromagnetic) material and be repulsed from a slab of diamagnetic material, no matter what its orientation is.

Both cases can be understood conceptually. Simply replace the loop with a bar magnet, draw its image and then see what the forces should be.