Jackson 5.17 Homework Problem Solution

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PROBLEM:

A current distribution **J**(**x**) exists in a medium of unit relative permeability adjacent to a semi-infinite slab of material having relative permeability μ_r and filling the half-space, $z < 0$.

(a) Show that for $z > 0$ the magnetic induction can be calculated by replacing the medium of permeability μ_r by an image current distribution **J***, with components,

$$\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_x(x, y, -z) \quad , \quad \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y(x, y, -z) \quad , \quad -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_z(x, y, -z)$$

(b) Show that for z < 0 the magnetic induction appears to be due to a current distribution $[2\mu_r/(\mu_r + 1)]\mathbf{J}$ in a medium of unit relative permeability.

SOLUTION:

(a) Using the method of images, we replace the effects of the actual currents on the material interface with an image current deep within the magnetic material. We will label the original current distribution **J**(**x**) and the image current distribution as **J***(**x**), where **J** is known but its image at this point needs to be determined. Because the mirror surface is a flat plane, we can safely assume that a piece of current in **J** at (x, y, z) will be mirrored by a piece of current in **J*** at (x, y, $-z$). Therefore:

$$J_i^*(x, y, z) = A_i J_i(x, y, -z) \quad \text{for each component, so that } i = x, y, z$$

The magnetic **B** field for $z > 0$ created by these currents are:

$$\mathbf{B}_{z>0}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{(\mathbf{J}(\mathbf{x}') + \mathbf{J}^*(\mathbf{x}')) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3\mathbf{x}'$$

For the $z < 0$ region, place an additional image current distribution $\mathbf{J}^{**} = a\,\mathbf{J}$ in its positive z location, which creates the field:

$$\mathbf{B}_{z<0}(\mathbf{x}) = \frac{\mu_0 \mu_r a}{4\pi} \int \frac{(\mathbf{J}(\mathbf{x}')) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3\mathbf{x}'$$

Now apply boundary conditions:

$$\left[(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0\right]_{\text{on S}}$$

$$\left[\mathbf{B}_{z>0} \cdot \hat{\mathbf{z}} = \mathbf{B}_{z<0} \cdot \hat{\mathbf{z}}\right]_{z=0}$$

$$\left[\frac{\mu_0}{4\pi}\int\frac{(\mathbf{J}(\mathbf{x}')+\mathbf{J}^*(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}'\right]_{z=0}\cdot\hat{\mathbf{z}}=\left[\frac{\mu_0\mu_r a}{4\pi}\int\frac{(\mathbf{J}(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}'\right]_{z=0}\cdot\hat{\mathbf{z}}$$

Because each piece mirrors each other piece, and because we are on the z = 0 plane, which is equidistant from each current piece and its image, the integrands must be equal.

$$\left[\frac{(\mathbf{J}(\mathbf{x}')+\mathbf{J}^*(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]_{z=0}\cdot\hat{\mathbf{z}}=\left[\mu_r a\frac{(\mathbf{J}(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]_{z=0}\cdot\hat{\mathbf{z}}$$

$$\left[(\mathbf{J}(\mathbf{x}')+\mathbf{J}^*(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')\right]_{z=0}\cdot\hat{\mathbf{z}}=\left[\mu_r a(\mathbf{J}(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')\right]_{z=0}\cdot\hat{\mathbf{z}}$$

If we break each current into a z component and transverse component, $\mathbf{J}=J_z\hat{\mathbf{z}}+\mathbf{J}_t$, we have:

$$\left[(J_z\hat{\mathbf{z}}+\mathbf{J}_t+J_z^*\hat{\mathbf{z}}+\mathbf{J}_t^*)\times(\mathbf{x}-\mathbf{x}')\right]_{z=0}\cdot\hat{\mathbf{z}}=\left[\mu_r a(J_z\hat{\mathbf{z}}+\mathbf{J}_t)\times(\mathbf{x}-\mathbf{x}')\right]_{z=0}\cdot\hat{\mathbf{z}}$$

Use the identity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$

$$\left[(\hat{\mathbf{z}}\times(J_z\hat{\mathbf{z}}+\mathbf{J}_t+J_z^*\hat{\mathbf{z}}+\mathbf{J}_t^*))\cdot(\mathbf{x}-\mathbf{x}')=\mu_r a(\hat{\mathbf{z}}\times(J_z\hat{\mathbf{z}}+\mathbf{J}_t))\cdot(\mathbf{x}-\mathbf{x}')\right]_{z=0}$$

$$\left[((\hat{\mathbf{z}}\times\mathbf{J}_t+\hat{\mathbf{z}}\times\mathbf{J}_t^*))\cdot(\mathbf{x}-\mathbf{x}')=\mu_r a((\hat{\mathbf{z}}\times\mathbf{J}_t))\cdot(\mathbf{x}-\mathbf{x}')\right]_{z=0}$$

$$\left[(J_x\hat{\mathbf{j}}-J_y\hat{\mathbf{i}}+J_x^*\hat{\mathbf{j}}-J_y^*\hat{\mathbf{i}})\cdot(\mathbf{x}-\mathbf{x}')=\mu_r a(J_x\hat{\mathbf{j}}-J_y\hat{\mathbf{i}})\cdot(\mathbf{x}-\mathbf{x}')\right]_{z=0}$$

$$(-J_y-J_y^*)(x-x')+(J_x+J_x^*)(y-y')=-\mu_r a J_y(x-x')+\mu_r a J_x(y-y')$$

This must be true for all x and y, so they must be independent:

$$J_y+J_y^*=\mu_r a J_y \quad , \quad J_x+J_x^*=\mu_r a J_x$$

$$\boxed{J_y^*=(\mu_r a-1)J_y} \quad , \quad \boxed{J_x^*=(\mu_r a-1)J_x}$$

Now apply the other boundary condition:

$$\left[\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}\right]_{\text{on S}}$$

There is no free current on the interface, so $\mathbf{K}$ = 0.

$$\left[\frac{1}{\mu_0}\hat{\mathbf{z}}\times\mathbf{B}_{z>0}=\frac{1}{\mu_r\mu_0}\hat{\mathbf{z}}\times\mathbf{B}_{z<0}\right]_{z=0}$$

$$\left[\frac{1}{\mu_0}\hat{\mathbf{z}}\times\left(\frac{\mu_0}{4\pi}\int\frac{(\mathbf{J}(\mathbf{x}')+\mathbf{J}^*(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}'\right)=\frac{1}{\mu_r\mu_0}\hat{\mathbf{z}}\times\left(\frac{\mu_0\mu_r a}{4\pi}\int\frac{(\mathbf{J}(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}'\right)\right]_{z=0}$$

$$\left[\hat{\mathbf{z}}\times\left((\mathbf{J}(\mathbf{x}')+\mathbf{J}^*(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')\right)=\hat{\mathbf{z}}\times\left(a(\mathbf{J}(\mathbf{x}'))\times(\mathbf{x}-\mathbf{x}')\right)\right]_{z=0}$$

Use the identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ being careful to remember that z' becomes $-z'$ when multiplied with **J***

$$(-z')(\mathbf{J}-\mathbf{J}^*)-(J_z+J_z^*)(\mathbf{x}-\mathbf{x}')=-z'a\mathbf{J}-aJ_z(\mathbf{x}-\mathbf{x}')$$

Expand everything into components and recognize that components as well as functional pieces are all independent.

$$\boxed{J_z^*=(a-1)J_z}\ ,\ \boxed{J_x^*=(1-a)J_x}\ ,\ \boxed{J_y^*=(1-a)J_y}\ ,\ \boxed{J_z^*=(a-1)J_z}$$

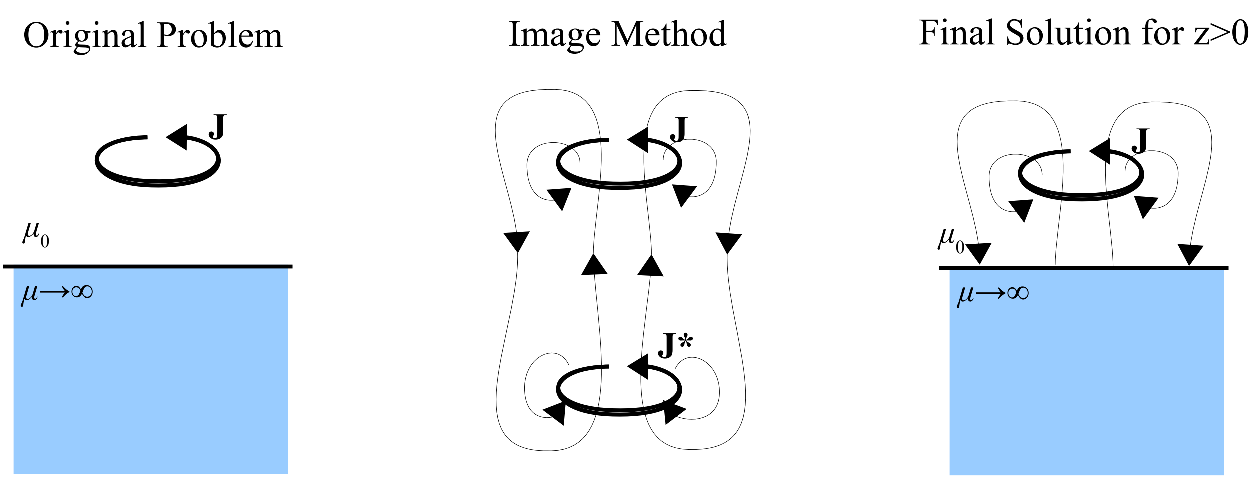
Now solve for a using any two appropriate equations from the boxed ones above:

$$\boxed{a=\frac{2}{\mu_r+1}}$$

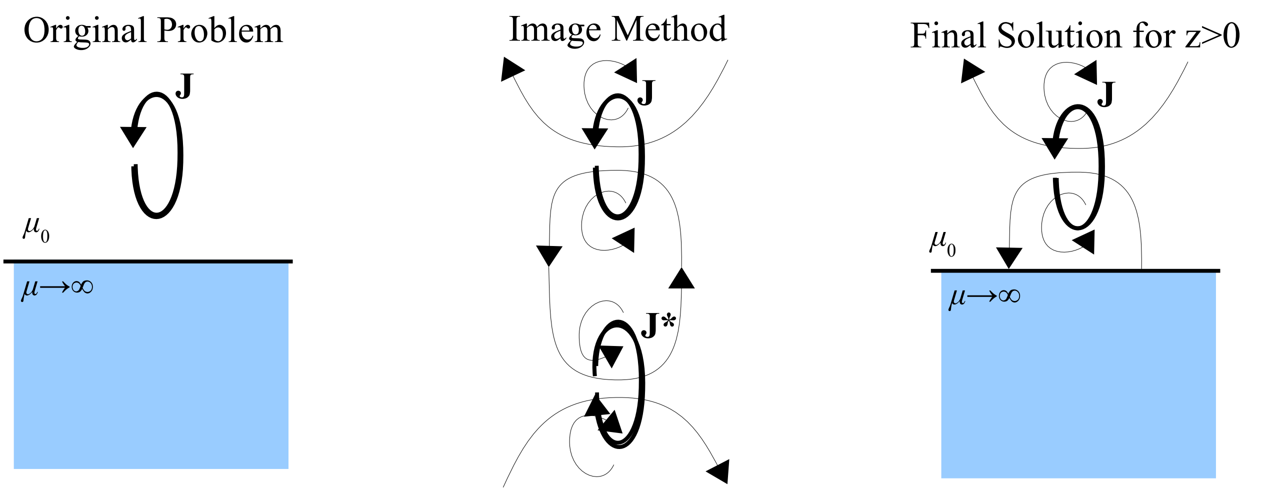
So that finally:

$$\boxed{J_x^*=\left(\frac{\mu_r-1}{\mu_r+1}\right)J_x(x,y,-z)}\ ,\ \boxed{J_y^*\left(\frac{\mu_r-1}{\mu_r+1}\right)J_y(x,y,-z)}\ ,\ \boxed{J_z^*=-\left(\frac{\mu_r-1}{\mu_r+1}\right)J_z(x,y,-z)}$$

What does this mean physically? If the material is a very strong paramagnetic material, than the relative permeability is effectively infinite. In this special case, the image current equals the original current in magnitude and loops in the same direction in the *x-y* directions, but in the opposite direction in the *z* direction:

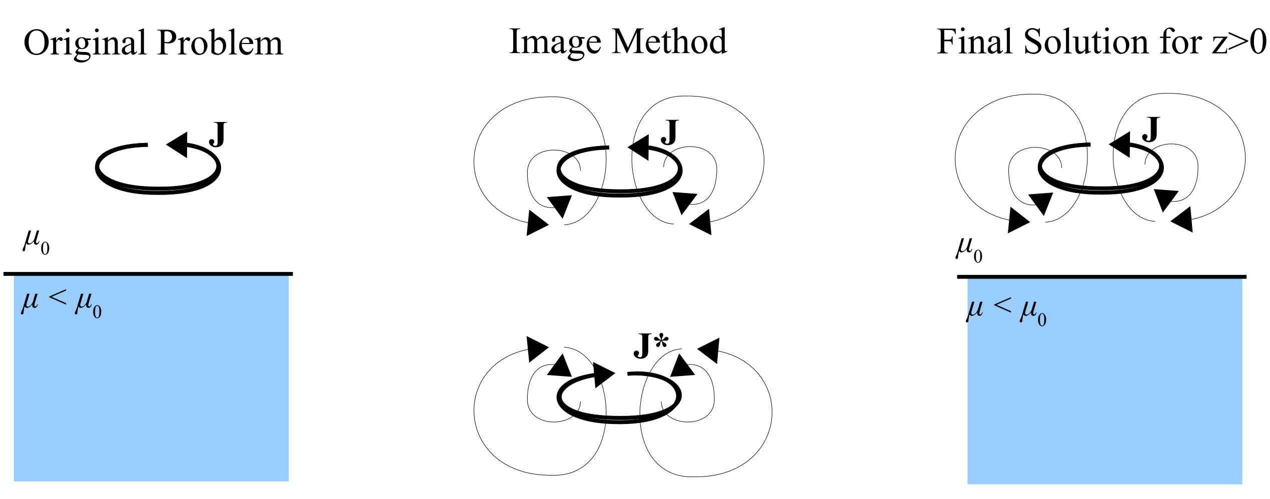


Paramagnetic materials tend to suck in magnetic field lines, no matter the shape or orientation of the current distribution. This means that paramagnetic materials attract permanent magnets and current distributions, no matter their orientation. A small loop of current like shown in the diagram above can be thought of as a small magnetic dipole, with North facing up. The image current can also be thought of as a small dipole magnet with North facing up. The South end of the top magnet is therefore facing the North end of the bottom magnet, so they will be attracted. Let us turn the loop of current to see this effect and to see the *z* component's behavior:



For a weak paramagnetic material, the same patterns result, but the overall effect is less.

For a diamagnetic material, $\mu_r < 1$, a quick glance at our equations shows us that the image current will have the *x* and *y* components going in the opposite direction as the original current, but the *z* component will be going in the same direction. Note that diamagnetic materials are typically very weak.



As we see, diamagnetic materials tend to repel magnetic field lines, current distributions, and permanent magnets, no matter their orientation.

(b) As found above, the field in the lower region is:

$$\mathbf{B}_{z<0}(\mathbf{x}) = \frac{\mu_0 \mu_r a}{4\pi} \int \frac{(\mathbf{J}(\mathbf{x}')) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3\mathbf{x}'$$

$$\mathbf{B}_{z<0}(\mathbf{x}) = \frac{\mu_0 \mu_r}{4\pi} \left(\frac{2}{\mu_r + 1} \right) \int \frac{(\mathbf{J}(\mathbf{x}')) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3 \mathbf{x}'$$

If we combine the extra factors with the current, it looks like the field is due to an effective current in a medium of unit permeability.

$$\mathbf{B}_{z<0}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{(\mathbf{J}_{\text{eff}}(\mathbf{x}')) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3 \mathbf{x}' \text{ where } \mathbf{J}_{\text{eff}} = \left(\frac{2\mu_r}{\mu_r + 1} \right) \mathbf{J}$$