# Jackson 5.16 Homework Problem Solution

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**PROBLEM:**

A circular loop of wire of radius $a$ and negligible thickness carries a current $I$. The loop is centered in a spherical cavity of radius $b > a$ in a large block of soft iron. Assume that the relative permeability of the iron is effectively infinite and that of the medium in the cavity, unity.

(a) In the approximation, of $b >> a$, show that the magnetic field at the center of the loop is augmented by a factor $(1 + a^3/2b^3)$ by the presence of the iron.

(b) What is the radius of the "image" current loop (carrying the same current) that simulates the effect of the iron for $r < b$?

**SOLUTION:**

We can find the magnetic vector potential **A** due to the loop by itself, than add in an extra arbitrary potential due to iron and apply boundary conditions to get the unique solution.

Use the Biot-Savart law and plug in the current density $\mathbf{J}=\hat{\boldsymbol{\phi}} I\,\delta(\theta-\pi/2)\,\delta(r-a)/a$:

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\hat{\boldsymbol{\phi}}' I\,\delta(\theta'-\pi/2)\,\delta(r'-a)/a}{|\mathbf{x}-\mathbf{x}'|}r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'$$

Expand the denominator into spherical harmonics.

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\hat{\boldsymbol{\phi}}' I\,\delta(\theta'-\pi/2)\,\delta(r'-a)/a\,4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'$$

Evaluate the deltas

$$\mathbf{A}=\mu_0 I a\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\int_0^{2\pi}\hat{\boldsymbol{\phi}}' Y_{lm}^*(\pi/2,\phi')Y_{lm}(\theta,\phi)\,d\phi'$$

where the smaller or bigger $r$ is now with respect to $r$ and $a$

Expand out the azimuthal unit vector according to $\hat{\boldsymbol{\phi}}'=-\sin\phi'\,\hat{\mathbf{i}}+\cos\phi'\,\hat{\mathbf{j}}$ and the spherical harmonics

$$\mathbf{A}=\frac{\mu_0 I a}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}\frac{r_<^l}{r_>^{l+1}}P_l^m(0)P_l^m(\cos\theta)e^{im\phi}\int_0^{2\pi}[-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}}]e^{-im\phi'}d\phi'$$

$$\mathbf{A}=\frac{\mu_0 I a}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}\frac{r_<^l}{r_>^{l+1}}P_l^m(0)P_l^m(\cos\theta)e^{im\phi}\int_0^{2\pi}[-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}}][\cos(m\phi')-i\sin(m\phi')]d\phi'$$

$$\mathbf{A}=\frac{\mu_0 I a}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}\frac{r_<^l}{r_>^{l+1}}P_l^m(0)P_l^m(\cos\theta)e^{im\phi}$$

$$\left[-\hat{\mathbf{i}}\int_0^{2\pi}\sin\phi'\cos(m\phi')d\phi'+\hat{\mathbf{i}}i\int_0^{2\pi}\sin\phi'\sin(m\phi')d\phi'+\hat{\mathbf{j}}\int_0^{2\pi}\cos\phi'\cos(m\phi')d\phi'-\hat{\mathbf{j}}i\int_0^{2\pi}\cos\phi'\sin(m\phi')d\phi'\right]$$

Due to orthogonality, $m$ must equal positive or negative one and we can do the integrals

$$\mathbf{A}=\frac{\mu_0 I a}{4\pi}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left[\frac{(l+1)!}{(l-1)!}P_l^{-1}(0)P_l^{-1}(\cos\theta)e^{-i\phi}\left[-\hat{\mathbf{i}}i\pi+\hat{\mathbf{j}}\pi\right]+\frac{(l-1)!}{(l+1)!}P_l^1(0)P_l^1(\cos\theta)e^{i\phi}\left[+\hat{\mathbf{i}}i\pi+\hat{\mathbf{j}}\pi\right]\right]$$

Now use the relation $P_l^{-1}=-\frac{(l-1)!}{(l+1)!}P_l^1$

$$\mathbf{A}=\frac{\mu_0 I a}{2}\sum_{l=0}^{\infty}\frac{(l-1)!}{(l+1)!}\frac{r_<^l}{r_>^{l+1}}P_l^1(0)P_l^1(\cos\theta)[-\sin\phi\,\hat{\mathbf{i}}+\cos\phi\,\hat{\mathbf{j}}]$$

Expand out the factorials enough to realize that things cancel out

$$\mathbf{A}=\frac{\mu_0 I a}{2}\sum_{l=0}^{\infty}\frac{1}{l(l+1)}\frac{r_<^l}{r_>^{l+1}}P_l^1(0)P_l^1(\cos\theta)\hat{\boldsymbol{\phi}}$$

For points inside the loop this becomes:

$$\boxed{\mathbf{A}=\frac{\mu_0 I}{2}\sum_{l=0}^{\infty}\frac{1}{l(l+1)}\frac{r^l}{a^l}P_l^1(0)P_l^1(\cos\theta)\hat{\boldsymbol{\phi}}}$$

Now if we put this loop in the middle of the cavity in the iron, we can add in another general potential to take into account the effects of the iron.

$$\mathbf{A}_{\text{iron}}=\sum_{l=0}^{\infty}A_l r^l P_l^1(\cos\theta)\hat{\boldsymbol{\phi}}$$

$$\mathbf{A}=\frac{\mu_0 I a}{2}\sum_{l=0}^{\infty}\frac{1}{l(l+1)}\frac{r_<^l}{r_>^{l+1}}P_l^1(0)P_l^1(\cos\theta)\hat{\boldsymbol{\phi}}+\sum_{l=0}^{\infty}A_l r^l P_l^1(\cos\theta)\hat{\boldsymbol{\phi}}$$

Now apply the boundary condition. The relative permeability of the iron is effectively infinite so that the components of the magnetic field **B** that are parallel to the sphere's surface at $r = b$ must vanish:

$$B_\theta(r=b)=0$$

$$0=[\hat{\boldsymbol{\theta}}\cdot\mathbf{B}]_{r=b}$$

$$0=[\hat{\boldsymbol{\theta}}\cdot(\nabla\times\mathbf{A})]_{r=b}$$

$$0=\left[-\frac{1}{r}\frac{\partial}{\partial r}(r\,A_\phi)\right]_{r=b}$$

$$0=\left[-\frac{1}{r}\frac{\partial}{\partial r}(r(\frac{\mu_0 I\,a}{2}\sum_{l=0}^{\infty}\frac{1}{l(l+1)}\frac{a^l}{r^{l+1}}P_l^1(0)P_l^1(\cos\theta)+\sum_{l=0}^{\infty}A_l r^l P_l^1(\cos\theta)))\right]_{r=b}$$

$$0=\sum_{l=0}^{\infty}\left[\frac{\mu_0 I\,a}{2}\frac{-1}{(l+1)}\frac{a^l}{b^{l+2}}P_l^1(0)+A_l(l+1)b^{l-1}\right]P_l^1(\cos\theta)$$

The Legendre functions are orthogonal, so each coefficient must vanish separately

$$A_l=\frac{\mu_0 I\,a}{2}\frac{1}{(l+1)^2}\frac{a^l}{b^{2l+1}}P_l^1(0)$$

The final solution then becomes:

$$\mathbf{A}=\frac{\mu_0 I\,a}{2}\sum_{l=0}^{\infty}\frac{1}{l+1}\left[\frac{1}{l}\frac{r_<^l}{r_>^{l+1}}+\frac{1}{l+1}\frac{a^l}{b^l}\frac{r^l}{b^{l+1}}\right]P_l^1(0)P_l^1(\cos\theta)\hat{\boldsymbol{\phi}}$$

For points inside the loop, $r < a$ this becomes

$$\mathbf{A}=\frac{\mu_0 I}{2}\sum_{l=0}^{\infty}\frac{1}{l(l+1)}\left[1+\frac{l}{l+1}\left(\frac{a}{b}\right)^{2l+1}\right]\frac{r^l}{a^l}P_l^1(0)P_l^1(\cos\theta)\hat{\boldsymbol{\phi}}$$

Comparing this to the solution for when no iron was present (the boxed equation above) we see that each term is augmented by a constant factor:

$$1+\frac{l}{l+1}\left(\frac{a}{b}\right)^{2l+1}$$

For $b >> a$ we have $(a/b) << 1$ so that higher powers of $(a/b)$ are negligible. The $l = 1$ will dominate and the augmentation factor becomes:

$$\boxed{1+\frac{1}{2}\left(\frac{a}{b}\right)^3}$$

Because this is a constant factor it can be moved out of derivatives. This means that the magnetic field is augmented by this same factor with which the vector potential is augmented.

(b) What is the radius of the "image" current loop (carrying the same current) that simulates the effect of the iron for $r < b$?

If there were an image loop of radius $c$ so that $c > b$ and current $I$, then the total potential inside the ring due to this loop and the original loop becomes:

$$\mathbf{A}=\frac{\mu_0 I}{2}\sum_{l=0}^{\infty}\frac{1}{l(l+1)}\left[\frac{1}{a^l}+\frac{1}{c^l}\right]r^l P_l^1(0)P_l^1(\cos\theta)\hat{\boldsymbol{\phi}}$$

Equate this to the potential for the ring in the iron cavity and equate terms:

$$c=\left[\frac{l+1}{l}\frac{b^{2l+1}}{a^{l+1}}\right]^{1/l}$$

Apparently, the radius is a function of $l$, which does not make sense for a constant radius. We cannot exactly model this problem using an image current. But for the special case of $b >> a$, we remember that only the $l = 1$ term contributes.

$$\boxed{c=2\frac{b^3}{a^2}}$$