Jackson 5.15 Homework Problem Solution

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PROBLEM:

Consider two long, straight wires, parallel to the z axis, spaced a distance d apart and carrying currents I in opposite directions. Describe the magnetic field $\mathbf{H}$ in terms of a magnetic scalar potential Φ_M, with $\mathbf{H}=-\nabla\Phi_M$.

(a) If the wires are parallel to the z axis with positions, $x = \pm d/2$, $y = 0$, show that in the limit of small spacing, the potential is approximately that of a two-dimensional dipole,

$$\Phi_M \approx -\frac{I\, d \sin\phi}{2\pi\rho} + O(d^2/\rho^2)$$

where ρ and ϕ are the usual polar coordinates.

(b) The closely spaced wires are now centered in a hollow right circular cylinder of steel, of inner (outer) radius a (b) and magnetic permeability $\mu = \mu_r\mu_0$. Determine the magnetic scalar potential in the three regions, $0 < \rho < a$, $a < \rho < b$, and $\rho > b$. Show that the field outside the steel cylinder is a two-dimensional dipole field, as in part a, but with strength reduced by the factor

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}$$

Relate your result to Problem 5.14.

(c) Assuming that $\mu_r >> 1$, and $b = a + t$, where the thickness $t << b$, write down an approximate expression for F and determine its numerical value for $\mu_r = 200$ (typical of steel at 20 G), $b = 1.25$ cm, $t = 3$ mm. The shielding effect is relevant for reduction of stray fields in residential and commercial 60 Hz, 110 or 220 V wiring. The figure illustrates the shielding effect for $a/b = 0.9$, $\mu_r = 100$.

SOLUTION:

First consider a single wire at $x = 0$, $y = 0$ carrying a current I in the positive z direction. Draw an Amperian loop around the wire, and due to symmetry, the field can come out.

$$\nabla\times\mathbf{H} = \mathbf{J}$$

$$\int_S (\nabla\times\mathbf{H})\cdot\hat{\mathbf{n}}\, da = \int_S \mathbf{J}\cdot\hat{\mathbf{n}}\, da$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I_{\text{enc}}$$

$$\mathbf{H} = \frac{I}{2\pi\rho}\hat{\phi}$$

Write this in terms of a scalar potential and integrate:

$$-\nabla\Phi_M = \frac{I}{2\pi\rho}\hat{\phi}$$

$$\hat{\phi}\frac{1}{\rho}\frac{\partial\Phi_M}{\partial\phi} = -\frac{I}{2\pi\rho}\hat{\phi}$$

$$\Phi_M = -\frac{I}{2\pi}\phi$$

Now shift this to $x = d/2$ and superpose another wire's potential at $x = -d/2$ with current going in the opposite direction.

$$\Phi_M = -\frac{I}{2\pi}\tan^{-1}\left(\frac{y}{x-d/2}\right) + \frac{I}{2\pi}\tan^{-1}\left(\frac{y}{x+d/2}\right)$$

$$\Phi_M = -\frac{I}{2\pi}\left[\tan^{-1}\left(\frac{\rho\sin\phi}{\rho\cos\phi - d/2}\right) + \tan^{-1}\left(\frac{-\rho\sin\phi}{\rho\cos\phi + d/2}\right)\right]$$

Use $\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left[\frac{A+B}{1-AB}\right]$

$$\Phi_M = -\frac{I}{2\pi}\tan^{-1}\left(\frac{\sin\phi\frac{d}{\rho}}{1-\frac{1}{4}\left(\frac{d}{\rho}\right)^2}\right)$$

Expand the arctangent in a Taylor series:

$$\tan^{-1}x = x - x^3/3 + \ldots$$

$$\Phi_M = -\frac{I}{2\pi}\left[\left(\frac{\sin\phi\frac{d}{\rho}}{1-\frac{1}{4}\left(\frac{d}{\rho}\right)^2}\right) - \left(\frac{\sin\phi\frac{d}{\rho}}{1-\frac{1}{4}\left(\frac{d}{\rho}\right)^2}\right)^3/3 + \ldots\right]$$

When $d \ll \rho$, all higher powers of d/ρ become negligible, leaving:

$$\boxed{\Phi_M = -\frac{I\,d\sin\phi}{2\pi\rho}}$$

(b) The closely spaced wires are now centered in a hollow right circular cylinder of steel, of inner (outer) radius a (b) and magnetic permeability $\mu = \mu_r\mu_0$. Determine the magnetic scalar potential in the three regions, $0 < \rho < a$, $a < \rho < b$, and $\rho > b$. Show that the field outside the steel cylinder is a two-dimensional dipole field, as in part a, but with strength reduced by the factor

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}$$

Relate your result to Problem 5.14.

In all regions, except near the origin, there is no free current and the material is linear and uniform. This means that the magnetic scalar potential in each region is the solution to Laplace's equations. We already know that the solution to the Laplace equation in polar coordinates is powers of ρ times sine functions of different arguments. Due to orthogonality therefore, only the $m = 1$ terms survive.

$$\Phi_{M,1} = \frac{Id}{2\pi}\sin\phi\left[-\frac{1}{\rho}+A\rho\right] \quad , \quad \Phi_{M,2} = \frac{Id}{2\pi}\sin\phi\left[B\frac{1}{\rho}+C\rho\right] \quad , \quad \Phi_{M,3} = \frac{Id}{2\pi}\sin\phi\left[D\frac{1}{\rho}\right]$$

where region 1 is the inner hollow core, region 2 is the steel, and region 3 is external space.

$$\mathbf{H} = -\nabla\Phi_M$$

$$\mathbf{H} = -\hat{\boldsymbol{\rho}}\frac{\partial\Phi_M}{\partial\rho} - \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\Phi_M}{\partial\phi}$$

$$\mathbf{H}_1 = \hat{\boldsymbol{\rho}}\left(\frac{Id}{2\pi}\sin\phi\left[-\frac{1}{\rho^2}-A\right]\right)+\hat{\boldsymbol{\phi}}\left(\frac{Id}{2\pi}\cos\phi\left[\frac{1}{\rho^2}-A\right]\right)$$

$$\mathbf{H}_2 = \hat{\boldsymbol{\rho}}\left(\frac{Id}{2\pi}\sin\phi\left[B\frac{1}{\rho^2}-C\right]\right)+\hat{\boldsymbol{\phi}}\left(\frac{Id}{2\pi}\cos\phi\left[-B\frac{1}{\rho^2}-C\right]\right)$$

$$\mathbf{H}_3 = \hat{\boldsymbol{\rho}}\left(\frac{Id}{2\pi}\sin\phi\left[D\frac{1}{\rho^2}\right]\right)+\hat{\boldsymbol{\phi}}\left(\frac{Id}{2\pi}\cos\phi\left[-D\frac{1}{\rho^2}\right]\right)$$

The boundary conditions that need to be satisfied are continuous tangential **H** fields and continuous normal **B** fields at each boundary.

$$\hat{\boldsymbol{\phi}}\cdot\mathbf{H}_1 = \hat{\boldsymbol{\phi}}\cdot\mathbf{H}_2 \text{ at } \rho=a \,,\ \hat{\boldsymbol{\phi}}\cdot\mathbf{H}_2 = \hat{\boldsymbol{\phi}}\cdot\mathbf{H}_3 \text{ at } \rho=b \,,\ \mu_0\hat{\boldsymbol{\rho}}\cdot\mathbf{H}_1 = \mu\hat{\boldsymbol{\rho}}\cdot\mathbf{H}_2 \text{ at } \rho=a \,,\ \mu\hat{\boldsymbol{\rho}}\cdot\mathbf{H}_2 = \mu_0\hat{\boldsymbol{\rho}}\cdot\mathbf{H}_3 \text{ at } \rho=b$$

$$Aa^2 - B - Ca^2 - 1 = 0 \quad , \quad B + Cb^2 - D = 0 \quad , \quad 1 + Aa^2 + \mu_r B - \mu_r C a^2 = 0 \quad , \quad \mu_r B - \mu_r C b^2 - D = 0$$

We have four equations and four unknowns, so we can find a unique solution.

$$A = \frac{-F}{2\mu_r a^2 b^2}\left((\mu_r+1)b^2 + (\mu_r-1)a^2\right) + \frac{1}{a^2}$$

$$B = -\frac{F}{2\mu_r}(\mu_r + 1)$$

$$C = -(\mu_r - 1)\frac{F}{2\mu_r b^2}$$

$$D = -F$$

where $F = \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}$

so that finally:

$$\Phi_{M,1} = -\frac{I\,d}{2\pi a}\sin\phi\left[\frac{a}{\rho} - \frac{\rho}{a} + \frac{\rho}{a}\left(\frac{F}{2\mu_r b^2}((\mu_r+1)b^2 + (\mu_r-1)a^2)\right)\right]$$
$$\Phi_{M,2} = -\frac{I\,d\,F}{4\pi b\mu_r}\sin\phi\left[(\mu_r+1)\frac{b}{\rho} + (\mu_r-1)\frac{\rho}{b}\right]$$
$$\Phi_{M,3} = F\left[\frac{-I\,d}{2\pi\rho}\sin\phi\right] \qquad \text{where} \quad F = \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}$$

There are several interesting points to these results. First of all, the potential in the outer region is just the potential of the original two-dimensional dipole, weakened in strength by a factor F. Inspection of F reveals that this effective dipole pattern in the outer region becomes weaker for thicker steel shells (b/a larger) and for shells with higher permeability. In the limit of infinite permeability of the shell, $F = 0$, meaning there are no magnetic fields outside the shell. The shell become a perfect magnetic shield.

The next interesting point is that inside the steel and in the hollow region, the field is an effective dipole field plus a uniform field.

(c) Assuming that $\mu_r >> 1$, and $b = a + t$, where the thickness $t << b$, write down an approximate expression for F and determine its numerical value for $\mu_r = 200$ (typical of steel at 20 G), $b = 1.25$ cm, $t = 3$ mm. The shielding effect is relevant for reduction of stray fields in residential and commercial 60 Hz, 110 or 220 V wiring. The figure illustrates the shielding effect for $a/b = 0.9$, $\mu_r = 100$.

With $a = b - t$ we have:

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}$$

$$F = \frac{4\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2\left(\frac{a}{b}\right)^2}$$

$$F = \frac{4\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2\left(1 - 2\left(\frac{t}{b}\right) + \left(\frac{t}{b}\right)^2\right)}$$

$$F = \frac{1}{1 + \frac{(\mu_r-1)^2}{4\mu_r}\left[2\left(\frac{t}{b}\right) - \left(\frac{t}{b}\right)^2\right]}$$

When $t << b$, $t/b << 1$, so that the last term can be dropped

$$F \approx \frac{1}{1 + \frac{(\mu_r-1)^2}{2\mu_r}\left(\frac{t}{b}\right)}$$

If $\mu_r >> 1$:

$$\boxed{F \approx \frac{1}{1 + \frac{\mu_r t}{2b}}}$$

$$F \approx \frac{1}{1 + \frac{(200)(0.3\,\text{cm})}{2(1.25\,\text{cm})}}$$

$F \approx 0.04$ (Note that using the expression without any approximations, $F = 0.0456$)