Jackson 5.14 Homework Problem Solution

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PROBLEM:

A long, hollow, right circular cylinder of inner (outer) radius a (b), and of relative permeability μ_r, is placed in a region of initially uniform magnetic-flux density $\mathbf{B}_0$ at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes of $\mathbf{B}$ on the cylinder axis to $\mathbf{B}_0$ as a function of $\log_{10} \mu_r$ for $a^2/b^2 = 0.5, 0.1$. Neglect end effects.

SOLUTION:

Align the axis of the cylinder with the z-axis and orient the original field to point in the positive x direction: $\mathbf{B}(\rho\to\infty)=B_0\hat{\mathbf{x}}$. Because the cylinder is long, we can neglect end effects and the problem becomes two-dimensional.

Separate the problem into three regions, In each region we have linear materials and no currents, so we can solve for the magnetic potential: (there *are* bound currents at the interfaces, but they only come into play when we apply boundary conditions if we split the problem into three regions)

$\nabla^2\Psi_M=0$ where $\mathbf{B}=-\nabla\Psi_M$ and the potential far away becomes $\Psi_M=-B_0\rho\cos\phi$

This is simply the Laplace equation in polar coordinates, for which we already know the general solution to be:

$$\Psi_M(\rho,\phi)=\sum_{m=1}^{\infty}(a_m\rho^m+b_m\rho^{-m})\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)$$

Outside the cylinder, apply the boundary condition at large ρ

$$-B_0\rho\cos\phi=\sum_{m=1}^{\infty}a_m\rho^m\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)$$

Due to orthogonality:

$$A_1=B_1=-B_0/(2a_1) \quad \text{and} \quad a_m=0 \quad \text{for } m>1$$

So that

$$\boxed{\Psi_{M,\text{out}}(\rho,\phi)=-B_0\rho\cos\phi+\sum_{m=1}^{\infty}\rho^{-m}\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)}$$

In the middle of the cylinder, $b>\rho>a$, we cannot apply any boundary conditions right away:

$$\boxed{\Psi_{M,\mathrm{mid}}(\rho,\phi)=\sum_{m=1}^{\infty}(c_m\rho^m+\rho^{-m})\left(C_m e^{im\phi}+D_m e^{-im\phi}\right)}$$

Inside the hollow center of the cylinder, $\rho < a$, the solution must be finite at the origin so that all negative m terms must go away.

$$\boxed{\Psi_{M,\mathrm{in}}(\rho,\phi)=\sum_{m=1}^{\infty}\rho^m\left(F_m e^{im\phi}+G_m e^{-im\phi}\right)}$$

There are no free currents and all materials are linear, so the boundary conditions become:

$$(\mathbf{B}_{\mathrm{out}}-\mathbf{B}_{\mathrm{mid}})\cdot\hat{\boldsymbol{\rho}}=0 \quad \text{and} \quad \left(\mathbf{B}_{\mathrm{out}}-\frac{1}{\mu_r}\mathbf{B}_{\mathrm{mid}}\right)\cdot\hat{\boldsymbol{\phi}}=0 \quad \text{at } \rho=b$$

$$(\mathbf{B}_{\mathrm{mid}}-\mathbf{B}_{\mathrm{in}})\cdot\hat{\boldsymbol{\rho}}=0 \quad \text{and} \quad \left(\frac{1}{\mu_r}\mathbf{B}_{\mathrm{mid}}-\mathbf{B}_{\mathrm{in}}\right)\cdot\hat{\boldsymbol{\phi}}=0 \quad \text{at } \rho=a$$

Substituting in our definition of the **B** field in terms of the scalar potential, these boundary conditions become:

$$\frac{\partial\Psi_{M,\mathrm{out}}}{\partial\rho}=\frac{\partial\Psi_{M,\mathrm{mid}}}{\partial\rho} \quad \text{and} \quad \frac{\partial\Psi_{M,\mathrm{out}}}{\partial\phi}=\frac{1}{\mu_r}\frac{\partial\Psi_{M,\mathrm{mid}}}{\partial\phi} \quad \text{at } \rho=b$$

$$\frac{\partial\Psi_{M,\mathrm{mid}}}{\partial\rho}=\frac{\partial\Psi_{M,\mathrm{in}}}{\partial\rho} \quad \text{and} \quad \frac{1}{\mu_r}\frac{\partial\Psi_{M,\mathrm{mid}}}{\partial\phi}=\frac{\partial\Psi_{M,\mathrm{in}}}{\partial\phi} \quad \text{at } \rho=a$$

Applying the first boundary condition gives:

$$-B_0\cos\phi+\sum_{m=1}^{\infty}(-m)b^{-m-1}\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)=\sum_{m=1}^{\infty}(c_m m b^{m-1}-m b^{-m-1})\left(C_m e^{im\phi}+D_m e^{-im\phi}\right)$$

$\boxed{A_1/b^2+(c_1-b^{-2})C_1=-B_0/2}$ and $B_1=A_1$, $D_1=C_1$

$\boxed{A_m+(c_m b^{2m}-1)C_m=0}$ for $m>1$

$\boxed{B_m+(c_m b^{2m}-1)D_m=0}$ for $m>1$

Applying the second boundary condition gives:

$$B_0 b\sin\phi+\sum_{m=1}^{\infty}b^{-m}\left(A_m i m e^{im\phi}-i m B_m e^{-im\phi}\right)=\frac{1}{\mu_r}\sum_{m=1}^{\infty}(c_m b^m+b^{-m})\left(C_m i m e^{im\phi}-i m D_m e^{-im\phi}\right)$$

$$\boxed{A_1/b^2-\frac{1}{\mu_r}(c_1+b^{-2})C_1=B_0/2}$$

$$\boxed{A_m - \frac{1}{\mu_r}(c_m b^{2m}+1)\,C_m = 0}\ \text{for } m > 1$$

$$\boxed{B_m - \frac{1}{\mu_r}(c_m b^{2m}+1)\,D_m = 0}\ \text{for } m > 1$$

Applying the third boundary condition gives:

$$\sum_{m=1}^{\infty}(c_m m a^{m-1} - m a^{-m-1})\left(C_m e^{im\phi} + D_m e^{-im\phi}\right) = \sum_{m=1}^{\infty} m a^{m-1}\left(F_m e^{im\phi} + G_m e^{-im\phi}\right)$$

$$\boxed{(c_1 - a^{-2})C_1 = F_1}\ \text{and } F_1 = G_1$$

$$\boxed{(c_m - a^{-2m})\,C_m = F_m}\ \text{for } m > 1$$

$$\boxed{(c_m - a^{-2m})\,D_m = G_m}\ \text{for } m > 1$$

Applying the last boundary condition gives:

$$\frac{1}{\mu_r}\sum_{m=1}^{\infty}(c_m a^m + a^{-m})\left(C_m i m e^{im\phi} - i m D_m e^{-im\phi}\right) = \sum_{m=1}^{\infty} a^m\left(F_m i m e^{im\phi} - i m G_m e^{-im\phi}\right)$$

$$\boxed{\frac{1}{\mu_r}(c_1 + a^{-2})C_1 = F_1}$$

$$\boxed{\frac{1}{\mu_r}(c_m + a^{-2m})\,C_m = F_m}\ \text{for } m > 1$$

$$\boxed{\frac{1}{\mu_r}(c_m + a^{-2m})\,D_m = G_m}\ \text{for } m > 1$$

Now we have several coupled equations and can solve for the unknowns. Trying to solve for the $m > 1$ cases, we soon find contradictory results, meaning the only possible solution is:

$A_m = 0$, $B_m = 0$, $c_m = 0$, $C_m = 0$, $D_m = 0$, $F_m = 0$, $G_m = 0$ for $m > 1$

All that is left is the $m = 1$ cases and we now solve for them:

$$\boxed{c_1 = -\frac{1}{a^2}\left(\frac{1+\mu_r}{1-\mu_r}\right)} \quad \boxed{F_1 = -2B_0\frac{b^2}{a^2}\frac{\mu_r}{\frac{b^2}{a^2}(1+\mu_r)^2 - (1-\mu_r)^2}} \quad F_1 = G_1, B_1 = A_1, D_1 = C_1$$

$$\boxed{C_1 = B_0 \frac{\mu_r(1-\mu_r)b^2}{\frac{b^2}{a^2}(1+\mu_r)^2-(1-\mu_r)^2}}$$

$$\boxed{A_1 = -B_0/2\, b^2(1-\mu_r^2)\frac{\frac{b^2}{a^2}-1}{\frac{b^2}{a^2}(1+\mu_r)^2-(1-\mu_r)^2}}$$

The final solution is then:

$$\boxed{\Psi_{M,\text{out}}(\rho,\phi) = -B_0\rho\cos\phi - \frac{(1-\mu_r^2)(b^2-a^2)}{2\mu_r a b} S\left(\frac{b}{\rho}\right) B_0 b\cos\phi}$$

$$\boxed{\Psi_{M,\text{mid}}(\rho,\phi) = -S\left[(1+\mu_r)\frac{\rho}{a}-(1-\mu_r)\frac{a}{\rho}\right] B_0 b\cos\phi}$$

where $S = \dfrac{2\mu_r a b}{b^2(1+\mu_r)^2-a^2(1-\mu_r)^2}$

$$\boxed{\Psi_{M,\text{in}}(\rho,\phi) = -\frac{b}{a}2 S B_0\rho\cos\phi}$$

Finally, using $\mathbf{B} = -\nabla\Psi_M$:

$$\mathbf{B} = -\hat{\boldsymbol{\rho}}\frac{\partial\Psi_M}{\partial\rho} - \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{d\Psi_M}{d\phi}$$

$$\boxed{\mathbf{B}_\text{out} = B_0\hat{\mathbf{x}} - \frac{(1-\mu_r^2)(b^2-a^2)}{2\mu_r a b} S\left(\frac{b^2}{\rho^2}\right) B_0[\hat{\mathbf{x}}+2\hat{\boldsymbol{\phi}}\sin\phi]}$$

$$\boxed{\mathbf{B}_\text{mid} = S B_0\frac{b}{a}(1+\mu_r)\hat{\mathbf{x}} + S B_0(1-\mu_r)\frac{a b}{\rho^2}[\hat{\mathbf{x}}+2\hat{\boldsymbol{\phi}}\sin\phi]}$$

where $S = \dfrac{2\mu_r a b}{b^2(1+\mu_r)^2-a^2(1-\mu_r)^2}$

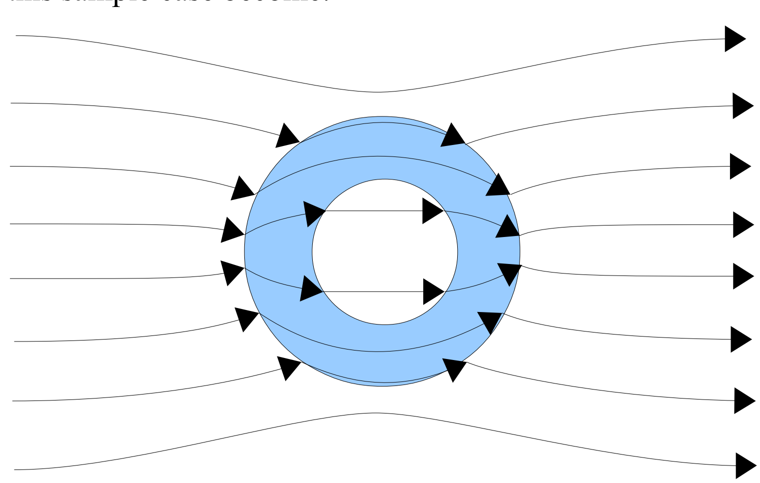
$$\boxed{\mathbf{B}_\text{in} = \frac{b}{a}2 S B_0\hat{\mathbf{x}}}$$

We can sketch a sample case to get an idea of what this solution looks like. Choose $b = 2a$ and a paramagnetic material, $\mu_r = 3$. The fields for this sample case become:

$$\mathbf{B}_\text{out} = B_0\hat{\mathbf{x}} + \frac{8}{5}\left(\frac{a}{\rho}\right)^2 B_0[\hat{\mathbf{x}}+2\hat{\boldsymbol{\phi}}\sin\phi]$$

$$\mathbf{B}_\text{mid} = \frac{8}{5}B_0\hat{\mathbf{x}} - \frac{4}{5}B_0\left(\frac{a}{\rho}\right)^2[\hat{\mathbf{x}}+2\hat{\boldsymbol{\phi}}\sin\phi]$$

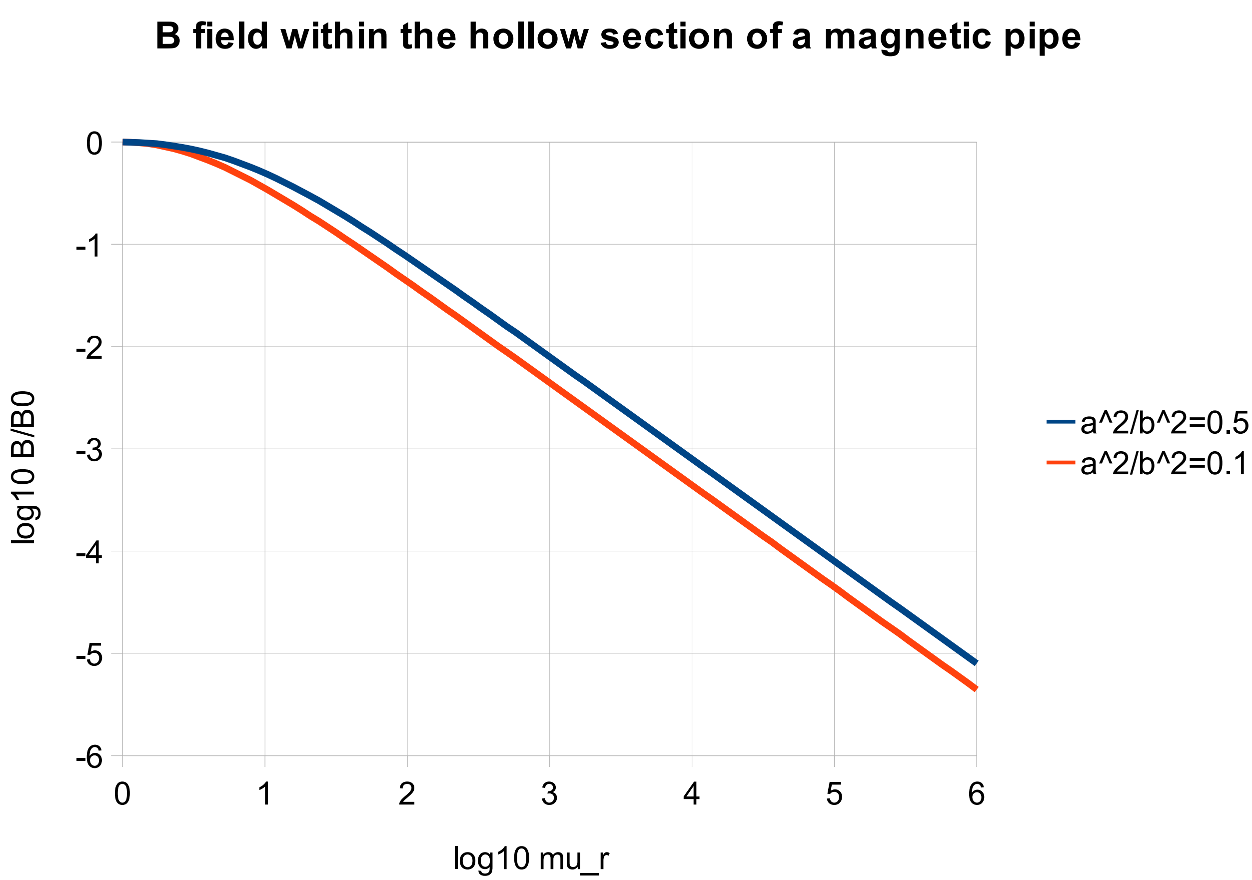
$$\mathbf{B}_\text{in} = \frac{4}{5}B_0\hat{\mathbf{x}}$$



We see that a hollow paramagnetic pipe
(and likewise ferromagnetic, like a steel pipe), tends to attract external field lines,
but then shields its hollow core from the fields.

We can sketch the magnitude of the **B** field inside the hollow core to get an idea of the shielding:

$$\boxed{\frac{B_{\text{in}}}{B_0}=\frac{4\mu_r}{(1+\mu_r)^2-\frac{a^2}{b^2}(1-\mu_r)^2}}$$



This is a log-log plot, so “-1” on the *y* scale means the field inside the pipe's core is 1/10th the strength of the externally applied field, “-2” means 1/100th the strength, and so on. We see that at $\mu_r = 1000$, common for some ferromagnets, the field has already been shielded to 1/100th of its original strength. The log-log plot also reveals that the behavior asymptotically approaches $B \propto 1/\mu_r$.