PROBLEM:
A long, hollow, right circular cylinder of inner (outer) radius \( a \) (\( b \)), and of relative permeability \( \mu_r \), is placed in a region of initially uniform magnetic-flux density \( B_0 \) at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes of \( B \) on the cylinder axis to \( B_0 \) as a function of \( \log_{10} \mu_r \) for \( a^2/b^2 = 0.5, 0.1 \). Neglect end effects.

SOLUTION:
Align the axis of the cylinder with the \( z \)-axis and orient the original field to point in the positive \( x \) direction: \( B(\rho \rightarrow \infty) = B_0 \hat{x} \). Because the cylinder is long, we can neglect end effects and the problem becomes two-dimensional.

Separate the problem into three regions. In each region we have linear materials and no currents, so we can solve for the magnetic potential: (there are bound currents at the interfaces, but they only come into play when we apply boundary conditions if we split the problem into three regions)

\[
\nabla^2 \Psi_M = 0 \text{ where } B = -\nabla \Psi_M \text{ and the potential far away becomes } \Psi_M = -B_0 \rho \cos \phi
\]

This is simply the Laplace equation in polar coordinates, for which we already know the general solution to be:

\[
\Psi_M(\rho, \phi) = \sum_{m=1}^{\infty} \left[ a_m \rho^m + b_m \rho^{-m} \right] \left( A_m e^{im\phi} + B_m e^{-im\phi} \right)
\]

Outside the cylinder, apply the boundary condition at large \( \rho \)

\[
-B_0 \rho \cos \phi = \sum_{m=1}^{\infty} a_m \rho^m \left( A_m e^{im\phi} + B_m e^{-im\phi} \right)
\]

Due to orthogonality:

\[
A_1 = B_1 = -B_0/(2a_1) \quad \text{and} \quad a_m = 0 \quad \text{for } m > 1
\]

So that

\[
\Psi_{M, \text{out}}(\rho, \phi) = -B_0 \rho \cos \phi + \sum_{m=1}^{\infty} \rho^{-m} \left( A_m e^{im\phi} + B_m e^{-im\phi} \right)
\]

In the middle of the cylinder, \( b > \rho > a \), we cannot apply any boundary conditions right away:
Inside the hollow center of the cylinder, \( \rho < a \), the solution must be finite at the origin so that all negative \( m \) terms must go away.

\[
\Psi_{M,\text{mid}}(\rho, \phi) = \sum_{m=1}^{\infty} \left( c_m \rho^m + \rho^{-m} \right) \left( C_m e^{im\phi} + D_m e^{-im\phi} \right)
\]

There are no free currents and all materials are linear, so the boundary conditions become:

\[
\begin{align*}
(B_{\text{out}} - B_{\text{mid}}) \cdot \hat{\rho} = 0 & \quad \text{and} \quad \left( B_{\text{out}} - \frac{1}{\mu_r} B_{\text{mid}} \right) \cdot \hat{\phi} = 0 \quad \text{at} \quad \rho = b \\
(B_{\text{mid}} - B_{\text{in}}) \cdot \hat{\rho} = 0 & \quad \text{and} \quad \left( \frac{1}{\mu_r} B_{\text{mid}} - B_{\text{in}} \right) \cdot \hat{\phi} = 0 \quad \text{at} \quad \rho = a
\end{align*}
\]

Substituting in our definition of the \( B \) field in terms of the scalar potential, these boundary conditions become:

\[
\begin{align*}
\frac{\partial \Psi_{M,\text{out}}}{\partial \rho} &= \frac{\partial \Psi_{M,\text{mid}}}{\partial \rho} \quad \text{and} \quad \frac{1}{\mu_r} \frac{\partial \Psi_{M,\text{out}}}{\partial \phi} = \frac{\partial \Psi_{M,\text{mid}}}{\partial \phi} \quad \text{at} \quad \rho = b \\
\frac{\partial \Psi_{M,\text{mid}}}{\partial \rho} &= \frac{\partial \Psi_{M,\text{in}}}{\partial \rho} \quad \text{and} \quad \frac{1}{\mu_r} \frac{\partial \Psi_{M,\text{mid}}}{\partial \phi} = \frac{\partial \Psi_{M,\text{in}}}{\partial \phi} \quad \text{at} \quad \rho = a
\end{align*}
\]

Applying the first boundary condition gives:

\[
\begin{align*}
-B_0 \cos \phi + \sum_{m=1}^{\infty} (-m) b^{-m-1} \left( A_m e^{im\phi} + B_m e^{-im\phi} \right) &= \sum_{m=1}^{\infty} \left( c_m m b^{-m} - m b^{-m-1} \right) \left( C_m e^{im\phi} + D_m e^{-im\phi} \right) \\
A_1 b^2 + (c_1 - b^{-2}) C_1 &= -B_0/2 \quad \text{and} \quad B_1 = A_1, D_1 = C_1 \\
A_m + (c_m b^{2m} - 1) C_m &= 0 \quad \text{for} \ m > 1 \\
B_m + (c_m b^{2m} - 1) D_m &= 0 \quad \text{for} \ m > 1
\end{align*}
\]

Applying the second boundary condition gives:

\[
\begin{align*}
B_0 b \sin \phi + \sum_{m=1}^{\infty} b^{-m} \left( A_m i m e^{im\phi} - i m B_m e^{-im\phi} \right) &= \frac{1}{\mu_r} \sum_{m=1}^{\infty} \left( c_m b^m + b^{-m} \right) \left( C_m i m e^{im\phi} - i m D_m e^{-im\phi} \right) \\
A_1 b^2 - \frac{1}{\mu_r} (c_1 + b^{-2}) C_1 &= B_0/2
\end{align*}
\]
Applying the third boundary condition gives:

\[
\sum_{m=1}^{\infty} (c_m a^{m-1} - m a^{-m}) \left( C_m e^{im\Phi} + D_m e^{-im\Phi} \right) = \sum_{m=1}^{\infty} m a^{m-1} \left( F_m e^{im\Phi} + G_m e^{-im\Phi} \right)
\]

\[(c_1 - a^{-2}) C_1 = F_1 \quad \text{and} \quad F_1 = G_1\]

\[(c_m - a^{-2m}) C_m = F_m \quad \text{for} \quad m > 1\]

\[(c_m - a^{-2m}) D_m = G_m \quad \text{for} \quad m > 1\]

Applying the last boundary condition gives:

\[
\frac{1}{\mu_r} \sum_{m=1}^{\infty} (c_m a^m + a^{-m}) \left( C_m i m e^{im\Phi} - i m D_m e^{-im\Phi} \right) = \sum_{m=1}^{\infty} a^{m} \left( F_m i m e^{im\Phi} - i m G_m e^{-im\Phi} \right)
\]

\[
\frac{1}{\mu_r} (c_1 + a^{-2}) C_1 = F_1
\]

\[
\frac{1}{\mu_r} (c_m + a^{-2m}) C_m = F_m \quad \text{for} \quad m > 1
\]

\[
\frac{1}{\mu_r} (c_m + a^{-2m}) D_m = G_m \quad \text{for} \quad m > 1
\]

Now we have several coupled equations and can solve for the unknowns. Trying to solve for the \( m > 1 \) cases, we soon find contradictory results, meaning the only possible solution is:

\[A_m = 0, B_m = 0, c_m = 0, C_m = 0, D_m = 0, F_m = 0, G_m = 0 \quad \text{for} \quad m > 1\]

All that is left is the \( m = 1 \) cases and we now solve for them:

\[
c_1 = -\frac{1}{a^2} \left( \frac{1+\mu_r}{1-\mu_r} \right)
\]

\[
F_1 = -2 B_0 \frac{b^2}{a^2} \frac{\mu_r}{b^2 (1+\mu_r)^2 - (1-\mu_r)^2}
\]

\[F_1 = G_1, B_1 = A_1, D_1 = C_1\]
The final solution is then:

\[
\Psi_{M,\text{out}}(\rho, \Phi) = -B_0 \rho \cos \Phi \frac{(1 - \mu_r^2) (b^2 - a^2)}{2 \mu_r a b} S \left( \frac{b}{\rho} \right) B_0 b \cos \Phi
\]

\[
\Psi_{M,\text{mid}}(\rho, \Phi) = -S \left[ (1 + \mu_r) \frac{\rho}{a} - (1 - \mu_r) \frac{a}{\rho} \right] B_0 b \cos \Phi
\]

\[
\Psi_{M,\text{in}}(\rho, \Phi) = -\frac{b}{a} 2 S B_0 \rho \cos \Phi
\]

Finally, using \( \mathbf{B} = -\nabla \Psi_M \):

\[
\mathbf{B} = -\rho \frac{\partial \Psi_M}{\partial \rho} - \hat{\Phi} \frac{1}{\rho} \frac{d \Psi_M}{d \Phi}
\]

\[
\mathbf{B}_{\text{out}} = B_0 \hat{\mathbf{x}} - \frac{(1 - \mu_r^2) (b^2 - a^2)}{2 \mu_r a b} S \left( \frac{b^2}{\rho^2} \right) B_0 [\hat{\mathbf{x}} + 2 \hat{\Phi} \sin \Phi]
\]

\[
\mathbf{B}_{\text{mid}} = S B_0 \frac{b}{a} (1 + \mu_r) \hat{\mathbf{x}} + S B_0 (1 - \mu_r) \frac{a b}{\rho^2} [\hat{\mathbf{x}} + 2 \hat{\Phi} \sin \Phi]
\]

\[
\mathbf{B}_{\text{in}} = \frac{b}{a} 2 S B_0 \hat{\mathbf{x}}
\]

We can sketch a sample case to get an idea of what this solution looks like. Choose \( b = 2a \) and a paramagnetic material, \( \mu_r = 3 \). The fields for this sample case become:

\[
\mathbf{B}_{\text{out}} = B_0 \hat{\mathbf{x}} + \frac{8}{5} \left( \frac{a}{\rho} \right)^2 B_0 [\hat{\mathbf{x}} + 2 \hat{\Phi} \sin \Phi]
\]

\[
\mathbf{B}_{\text{mid}} = \frac{8}{5} B_0 \hat{\mathbf{x}} - \frac{4}{5} B_0 \left( \frac{a}{\rho} \right)^2 [\hat{\mathbf{x}} + 2 \hat{\Phi} \sin \Phi]
\]

\[
\mathbf{B}_{\text{in}} = \frac{4}{5} B_0 \hat{\mathbf{x}}
\]
We see that a hollow paramagnetic pipe
(and likewise ferromagnetic, like a steel pipe), tends to attract external field lines,
but then shields its hollow core from the fields.

We can sketch the magnitude of the \( B \) field inside the hollow core to get an idea of the shielding:

\[
\frac{B_{in}}{B_0} = \frac{4\mu_r}{(1 + \mu_r)^2 - \frac{a^2}{b^2}(1 - \mu_r)^2}
\]

This is a log-log plot, so “-1” on the \( y \) scale means the field inside the pipe's core is 1/10\(^{th} \) the strength of the externally applied field, “-2” means 1/100\(^{th} \) the strength, and so on. We see that at \( \mu_r = 1000 \), common for some ferromagnets, the field has already been shielded to 1/100\(^{th} \) of its original strength. The log-log plot also reveals that the behavior asymptotically approaches \( B \sim 1/\mu_r \).