Jackson 5.13 Homework Problem Solution

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PROBLEM:

A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the sphere.

SOLUTION:

The current density in spherical coordinates is:

$$\mathbf{J}=\sigma\,\omega\,a\sin\theta\,\delta(r-a)\hat{\boldsymbol{\phi}}$$

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\sigma\,\omega\,a\sin\theta'\delta(r'-a)\hat{\boldsymbol{\phi}}'}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

Expand the denominator in spherical harmonics.

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\sigma\,\omega\,\delta(r'-a)\hat{\boldsymbol{\phi}}'4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)r'^2\sin^2\theta'dr'd\theta'd\phi'$$

We must be careful and realize that $\hat{\boldsymbol{\phi}}'\neq\hat{\boldsymbol{\phi}}$ because the primed variables are being integrated over. The best way to handle this is to use the expansion: $\hat{\boldsymbol{\phi}}'=-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}}$

$$\mathbf{A}=\mu_0\sigma\,\omega\,a\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\delta(r'-a)(-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}})\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{r'^{l-1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)\sin^2\theta'dr'd\theta'd\phi'$$

for $r<r'$ (inside the sphere).

$$\mathbf{A}=\mu_0\sigma\,\omega\,a\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\delta(r'-a)(-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}})\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r'^{l+2}}{r^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)\sin^2\theta'dr'd\theta'd\phi'$$

for $r>r'$ (outside the sphere).

Now evaluate the delta functions and rearrange:

$$\mathbf{A}=\mu_0\sigma\,\omega\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{a^{l-2}}Y_{lm}(\theta,\phi)I_{l,m}\quad\text{(inside)}$$

$$\mathbf{A}=\mu_0\sigma\,\omega\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{a^{l+3}}{r^{l+1}}Y_{lm}(\theta,\phi)I_{l,m}\quad\text{(outside)}$$

where $I_{l,m}=\int_0^{2\pi}\int_0^{\pi}(-\sin\phi'\,\hat{\mathbf{i}}+\cos\phi'\,\hat{\mathbf{j}})Y_{lm}^{*}(\theta',\phi')\sin^2\theta'\,d\theta'\,d\phi'$

Expand the definition of the spherical harmonics to solve the integrals:

$$I_{l,m}=\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\int_0^{2\pi}(-\sin\phi'\,\hat{\mathbf{i}}+\cos\phi'\,\hat{\mathbf{j}})e^{-im\phi'}d\phi'\int_0^{\pi}P_l^m(\cos\theta')\sin^2\theta'\,d\theta'$$

Due to the orthogonality in the first integral, all terms vanish except m = 1 and m = -1. The first integral can then be easily calculated to yield:

$$I_{l,\pm1}=\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l\mp1)!}{(l\pm1)!}}\pi(\pm i\,\hat{\mathbf{i}}+\hat{\mathbf{j}})\int_0^{\pi}P_l^{\pm1}(\cos\theta')\sin^2\theta'\,d\theta'$$

Make the substitution $x=\cos\theta$, $dx=-\sin\theta\,d\theta$ and recognize

$$\sin\theta=\sqrt{1-x^2}=P_1^1(x)=-2P_1^{-1}(x)$$

$$I_{l,1}=\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-1)!}{(l+1)!}}\pi(i\,\hat{\mathbf{i}}+\hat{\mathbf{j}})\int_{-1}^{1}P_l^1(x)P_1^1(x)\,dx$$

$$I_{l,-1}=\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+1)!}{(l-1)!}}\pi(-i\,\hat{\mathbf{i}}+\hat{\mathbf{j}})(-2)\int_{-1}^{1}P_l^{-1}(x)P_1^{-1}(x)dx$$

Due to orthogonality, only the l = 1 term is nonzero, leading to:

$$I_{1,\pm1}=\sqrt{\frac{2\pi}{3}}(i\,\hat{\mathbf{i}}\pm\hat{\mathbf{j}})$$

The solution is now just the sum of the l = 1, m = -1 term and the l = 1, m = 1 term.

$$\mathbf{A}=\frac{\mu_0\sigma\,\omega}{3}a\,r[Y_{1,1}(\theta,\phi)I_{1,1}+Y_{1,-1}(\theta,\phi)I_{1,-1}]\quad\text{(inside)}$$

$$\mathbf{A}=\frac{\mu_0\sigma\,\omega}{3}\frac{a^4}{r^2}[Y_{1,1}(\theta,\phi)I_{1,1}+Y_{1,-1}(\theta,\phi)I_{1,-1}]\quad\text{(outside)}$$

We note that $Y_{1,-1}=-Y_{1,1}^{*}$ and $I_{1,-1}=-I_{1,1}^{*}$ so that:

$$\mathbf{A}=\frac{\mu_0\sigma\,\omega}{3}a\,r[Y_{1,1}(\theta,\phi)I_{1,1}+Y_{1,1}^{*}(\theta,\phi)I_{1,1}^{*}]\quad\text{(inside)}$$

$$\mathbf{A}=\frac{\mu_0 \sigma \omega}{3} \frac{a^4}{r^2}[Y_{1,1}(\theta,\phi) I_{1,1}+Y_{1,1}^*(\theta,\phi) I_{1,1}^*] \quad \text{(outside)}$$

For complex numbers in general $z+z^*=2\Re(z)$ so that our final solutions become

$$\boxed{\mathbf{A}=\frac{\mu_0 \sigma \omega}{3} a\, r \sin\theta\, \hat{\boldsymbol{\phi}}} \quad \text{(inside)}$$

$$\boxed{\mathbf{A}=\frac{\mu_0 \sigma \omega}{3} \frac{a^4}{r^2} \sin\theta\, \hat{\boldsymbol{\phi}}} \quad \text{(outside)}$$

The magnetic flux density then becomes:

$$\mathbf{B}=\nabla\times\mathbf{A}$$

Expand in spherical coordinates, keeping only terms that are non-zero for this case:

$$\mathbf{B}=\hat{\mathbf{r}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi)-\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial r}(r\, A_\phi)$$

Inside:

$$\mathbf{B}=\frac{2}{3} a\, \mu_0 \sigma \omega \left[\cos\theta\, \hat{\mathbf{r}}-\sin\theta\, \hat{\boldsymbol{\theta}}\right]$$

$$\boxed{\mathbf{B}=\frac{2}{3} a\, \mu_0 \sigma \omega\, \hat{\mathbf{z}}}$$

Outside:

$$\mathbf{B}=\frac{\mu_0 \sigma \omega}{3} \frac{a^4}{r^3}\left[2\cos\theta\, \hat{\mathbf{r}}+\sin\theta\, \hat{\boldsymbol{\theta}}\right]$$

$$\boxed{\mathbf{B}=\frac{\mu_0 \sigma \omega}{3} \frac{a^4}{r^3}[3\cos\theta\, \hat{\mathbf{r}}-\hat{\mathbf{z}}]}$$