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Jackson 5.11 Homework Problem Solution

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PROBLEM:

A circular loop of wire carrying a current I is located with its center at the origin of coordinates and the normal to its plane having spherical angles θ_0, ϕ_0 . There is an applied magnetic field, $B_x = B_0(1 + \beta y)$ and $B_y = B_0(1 + \beta x)$.

(a) Calculate the force acting on the loop without making any approximations. Compare your result with the approximate result (5.69). Comment.

SOLUTION:

(a) We could brute force our way through this problem and soon discover that the math gets very messy very quickly. Let us instead apply simplifications as much as possible as we go. First of all, based on the symmetry, a circular current loop can never feel a net linear force from a spatially constant magnetic field. This is because all the force experienced by one half is canceled by the force on the other half. In so far as the linear force is concerned, the normalized magnetic field \mathbf{B}^n is therefore:

$$\mathbf{B}^n = y \hat{\mathbf{i}} + x \hat{\mathbf{j}} \quad \text{where} \quad \mathbf{B}^n = \frac{\mathbf{B}}{B_0 \beta}$$

The force on a wire is given by the equation:

$$\mathbf{F} = \int \mathbf{J} \times \mathbf{B} dV$$

The current density \mathbf{J} has a very simple expression in a rotated reference frame where the normal to the loop is in the z direction (frame R') and the magnetic field \mathbf{B} has a simple expression in the original reference frame (frame R). In the rotated frame, the current density is just:

$$\mathbf{J}(\mathbf{x}') = I \delta(r' - a) \frac{\delta(\theta' - \pi/2)}{r'} \hat{\boldsymbol{\phi}}'$$

The force in the rotated frame is therefore:

$$\mathbf{F}(\mathbf{x}') = \int \mathbf{J}(\mathbf{x}') \times \mathbf{B}(\mathbf{x}') dV$$

$$\mathbf{F}(\mathbf{x}') = \int_0^{2\pi} \int_0^\pi \int_0^\infty I \delta(r' - a) \frac{\delta(\theta' - \pi/2)}{r'} \hat{\boldsymbol{\phi}}' \times \mathbf{B}(\mathbf{x}') r'^2 \sin \theta' dr' d\theta' d\phi'$$

$$\mathbf{F}(\mathbf{x}') = a I \left[\int_0^{2\pi} \hat{\boldsymbol{\phi}}' \times \mathbf{B}(\mathbf{x}') d\phi' \right]_{r'=a, z'=0}$$

$$\mathbf{F}(\mathbf{x}') = a I B_0 \beta \left[\int_0^{2\pi} \hat{\boldsymbol{\phi}}' \times \mathbf{B}^n(\mathbf{x}') d\phi' \right]_{r'=a, z'=0}$$

$$\mathbf{F}(\mathbf{x}') = a I B_0 \beta \left[\int_0^{2\pi} (-\sin\phi' \hat{\mathbf{i}}' + \cos\phi' \hat{\mathbf{j}}') \times (B_x^n \hat{\mathbf{i}}' + B_y^n \hat{\mathbf{j}}' + B_z^n \hat{\mathbf{k}}') d\phi' \right]_{r'=a, z'=0}$$

$$\mathbf{F}(\mathbf{x}') = a I B_0 \beta \left[\int_0^{2\pi} [-\sin\phi' B_y^n \hat{\mathbf{k}}' - \cos\phi' \times B_x^n \hat{\mathbf{k}}' + \sin\phi' B_z^n \hat{\mathbf{j}}' + \cos\phi' \times B_z^n \hat{\mathbf{i}}'] d\phi' \right]_{r'=a, z'=0}$$

$$F_{x'} = a I B_0 \beta \left[\int_0^{2\pi} \cos\phi' B_z^n d\phi' \right]_{r'=a, z'=0}$$

$$F_{y'} = a I B_0 \beta \left[\int_0^{2\pi} \sin\phi' B_z^n d\phi' \right]_{r'=a, z'=0}$$

$$F_{z'} = -a I B_0 \beta \left[\int_0^{2\pi} [\sin\phi' B_y^n + \cos\phi' B_x^n] d\phi' \right]_{r'=a, z'=0}$$

Now we need to find the magnetic field in the rotated reference frame. We do this by rotating an angle $-\phi_0$ about the z axis to get the loop normal vector into the x - z plane, and then an angle $-\theta_0$ about the y axis to get the loop normal aligned with the z axis.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta_0 & 0 & -\sin\theta_0 \\ 0 & 1 & 0 \\ \sin\theta_0 & 0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} \cos\phi_0 & \sin\phi_0 & 0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta_0 \cos\phi_0 & \sin\phi_0 \cos\theta_0 & -\sin\theta_0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ \sin\theta_0 \cos\phi_0 & \sin\theta_0 \sin\phi_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}} \quad (1)$$

Because this matrix is orthogonal, its inverse is its transpose, leading to:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\phi_0 \cos\theta_0 & -\sin\phi_0 & \cos\phi_0 \sin\theta_0 \\ \sin\phi_0 \cos\theta_0 & \cos\phi_0 & \sin\phi_0 \sin\theta_0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Note that the only primed variable locations we need are on the loop, which are all at $z' = 0$. Also, the magnetic field does not have a z component or z functionality. The above matrix therefore reduces down to:

$$\boxed{x = \cos\phi_0 \cos\theta_0 x' - \sin\phi_0 y' \quad \text{and} \quad y = \sin\phi_0 \cos\theta_0 x' + \cos\phi_0 y'} \quad (2)$$

Now transform the normalized magnetic field components into the primed frame using the Eq. 1:

$$\begin{bmatrix} B_{x'}^n \\ B_{y'}^n \\ B_{z'}^n \end{bmatrix} = \begin{bmatrix} \cos \theta_0 \cos \phi_0 & \sin \phi_0 \cos \theta_0 & -\sin \theta_0 \\ -\sin \phi_0 & \cos \phi_0 & 0 \\ \sin \theta_0 \cos \phi_0 & \sin \theta_0 \sin \phi_0 & \cos \theta_0 \end{bmatrix} \begin{bmatrix} B_x^n \\ B_y^n \\ B_z^n \end{bmatrix}$$

$$\begin{bmatrix} B_{x'}^n \\ B_{y'}^n \\ B_{z'}^n \end{bmatrix} = \begin{bmatrix} \cos \theta_0 \cos \phi_0 & \sin \phi_0 \cos \theta_0 & -\sin \theta_0 \\ -\sin \phi_0 & \cos \phi_0 & 0 \\ \sin \theta_0 \cos \phi_0 & \sin \theta_0 \sin \phi_0 & \cos \theta_0 \end{bmatrix} \begin{bmatrix} y \\ x \\ 0 \end{bmatrix}$$

$$B_{x'}^n = \cos \theta_0 \cos \phi_0 y + \sin \phi_0 \cos \theta_0 x$$

$$B_{y'}^n = -\sin \phi_0 y + \cos \phi_0 x$$

$$B_{z'}^n = \sin \theta_0 \cos \phi_0 y + \sin \theta_0 \sin \phi_0 x$$

We have converted the vector components, but we still must convert the coordinate functionality. We use the Eq. 2 above to do this, substituting the unprimed coordinates with primed to find:

$$B_{x'}^n = \cos^2 \theta_0 \sin(2\phi_0) x' + \cos \theta_0 \cos(2\phi_0) y'$$

$$B_{y'}^n = \cos(2\phi_0) \cos \theta_0 x' - \sin(2\phi_0) y'$$

$$B_{z'}^n = \frac{1}{2} \sin(2\theta_0) \sin(2\phi_0) x' + \cos(2\phi_0) \sin \theta_0 y'$$

Finally, the integral is over the azimuthal angle, so we substitute: $x' = a \cos \phi'$ and $y' = a \sin \phi'$:

$$B_{x'}^n = \cos^2 \theta_0 \sin(2\phi_0) a \cos \phi' + \cos \theta_0 \cos(2\phi_0) a \sin \phi'$$

$$B_{y'}^n = \cos(2\phi_0) \cos \theta_0 a \cos \phi' - \sin(2\phi_0) a \sin \phi'$$

$$B_{z'}^n = \frac{1}{2} \sin(2\theta_0) \sin(2\phi_0) a \cos \phi' + \cos(2\phi_0) \sin \theta_0 a \sin \phi'$$

Now insert these into the force integrals:

$$\begin{aligned} F_{x'} &= a^2 I B_0 \beta \left[\frac{1}{2} \sin(2\theta_0) \sin(2\phi_0) \int_0^{2\pi} \cos^2 \phi' d\phi' + \cos(2\phi_0) \sin \theta_0 \int_0^{2\pi} \cos \phi' \sin \phi' d\phi' \right] \\ F_{y'} &= a^2 I B_0 \beta \left[\frac{1}{2} \sin(2\theta_0) \sin(2\phi_0) \int_0^{2\pi} \sin \phi' \cos \phi' d\phi' + \cos(2\phi_0) \sin \theta_0 \int_0^{2\pi} \sin^2 \phi' d\phi' \right] \\ F_{z'} &= -a^2 I B_0 \beta \left[2 \cos(2\phi_0) \cos \theta_0 \int_0^{2\pi} \sin \phi' \cos \phi' d\phi' - \sin(2\phi_0) \int_0^{2\pi} \sin^2 \phi' d\phi' + \cos^2 \theta_0 \sin(2\phi_0) \int_0^{2\pi} \cos^2 \phi' d\phi' \right] \end{aligned}$$

$$\text{Using } \int_0^{2\pi} \cos^2 \phi' d\phi' = \pi, \int_0^{2\pi} \sin^2 \phi' d\phi' = \pi, \text{ and } \int_0^{2\pi} \cos \phi' \sin \phi' d\phi' = 0$$

$$F_{x'} = a^2 I B_0 \beta \frac{\pi}{2} \sin(2\theta_0) \sin(2\phi_0)$$

$$F_{y'} = a^2 I B_0 \beta \pi \cos(2\phi_0) \sin\theta_0$$

$$F_{z'} = a^2 I B_0 \beta \pi \sin(2\phi_0) \sin^2\theta_0$$

Now we just need to transform back to the original coordinate system:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = a^2 I B_0 \beta \pi \begin{bmatrix} \cos\phi_0 \cos\theta_0 & -\sin\phi_0 & \cos\phi_0 \sin\theta_0 \\ \sin\phi_0 \cos\theta_0 & \cos\phi_0 & \sin\phi_0 \sin\theta_0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \sin(2\theta_0) \sin(2\phi_0) \\ \cos(2\phi_0) \sin\theta_0 \\ \sin(2\phi_0) \sin^2\theta_0 \end{bmatrix}$$

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = a^2 I B_0 \beta \pi \begin{bmatrix} \cos\phi_0 \cos\theta_0 \frac{1}{2} \sin(2\theta_0) \sin(2\phi_0) - \sin\phi_0 \cos(2\phi_0) \sin\theta_0 + \cos\phi_0 \sin\theta_0 \sin(2\phi_0) \sin^2\theta_0 \\ \sin\phi_0 \cos\theta_0 \frac{1}{2} \sin(2\theta_0) \sin(2\phi_0) + \cos\phi_0 \cos(2\phi_0) \sin\theta_0 + \sin\phi_0 \sin\theta_0 \sin(2\phi_0) \sin^2\theta_0 \\ -\sin\theta_0 \frac{1}{2} \sin(2\theta_0) \sin(2\phi_0) + \cos\theta_0 \sin(2\phi_0) \sin^2\theta_0 \end{bmatrix}$$

$$\begin{aligned} F_x &= a^2 I B_0 \beta \pi \sin\theta_0 \sin\phi_0 \\ F_y &= a^2 I B_0 \beta \pi \sin\theta_0 \cos\phi_0 \\ F_z &= 0 \end{aligned}$$

The approximate result is given by

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

$$\mathbf{m} = I \pi a^2 (\sin\theta_0 \cos\phi_0 \hat{\mathbf{i}} + \sin\theta_0 \sin\phi_0 \hat{\mathbf{j}} + \cos\theta_0 \hat{\mathbf{k}})$$

$$\mathbf{F} = I \pi a^2 B_0 \beta \nabla((\sin\theta_0 \cos\phi_0 \hat{\mathbf{i}} + \sin\theta_0 \sin\phi_0 \hat{\mathbf{j}} + \cos\theta_0 \hat{\mathbf{k}}) \cdot (y \hat{\mathbf{i}} + x \hat{\mathbf{j}}))$$

$$\mathbf{F} = I \pi a^2 B_0 \beta \nabla(\sin\theta_0 \cos\phi_0 y + \sin\theta_0 \sin\phi_0 x)$$

$$\begin{aligned} F_x &= a^2 I B_0 \beta \pi \sin\theta_0 \sin\phi_0 \\ F_y &= a^2 I B_0 \beta \pi \sin\theta_0 \cos\phi_0 \\ F_z &= 0 \end{aligned}$$

This is the exact same result as before.