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## Jackson 4.9 Homework Problem Solution

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### **PROBLEM:**

A point charge  $q$  is located in free space a distance  $d$  from the center of a dielectric sphere of radius  $a$  ( $a < d$ ) and dielectric constant  $\epsilon/\epsilon_0$ .

- Find the potential at all points in space as an expansion in spherical harmonics.
- Calculate the rectangular components of the electric field *near* the center of the sphere.
- Verify that, in the limit  $\epsilon/\epsilon_0 \rightarrow \infty$ , your result is the same as that for the conducting sphere.

### **SOLUTION:**

(a) Place the sphere at the origin and the point charge on the  $z$ -axis at  $z = d$ . Now the problem has azimuthal symmetry so that  $m = 0$  and the solution will involve only ordinary Legendre polynomials. There are two regions that we will have to solve separately and link by boundary conditions:

$\Phi_{\text{in}}$  exists for  $r < a$  where we have  $\epsilon$

$\Phi_{\text{out}}$  exists for  $r > a$  where we have  $\epsilon_0$

Let us solve first for the outside region first. Place an image charge  $q'$  at  $z = a^2/d$  to account for the effects of the dielectric sphere. (This location is suggested to us by the charge near a conducting sphere problem done earlier.) The potential in the outside region due to the point charge and image charge is:

$$\Phi_{\text{out}} = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{|\mathbf{r} - d\hat{\mathbf{z}}|} + \frac{q'}{|\mathbf{r} - (a^2/d)\hat{\mathbf{z}}|} \right)$$

We can expand each term into a series of Legendre polynomials and combine:

$$\frac{1}{|\mathbf{r} - \mathbf{r}_0|} = \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos\theta) \quad \text{where } r_{<} \text{ is the smaller of } (r, r_0)$$

$$\Phi_{\text{out}} = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} P_l(\cos\theta) \frac{1}{r^{l+1}} \left( qd^l + q' \frac{a^{2l}}{d^l} \right) \quad \text{if } r > d$$

$$\Phi_{\text{out}} = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} P_l(\cos\theta) \left( q \frac{r^l}{d^{l+1}} + q' \frac{a^{2l}}{d^l r^{l+1}} \right) \quad \text{if } r < d$$

Now let us solve for the potential inside the sphere. There is no charge inside the sphere, so all we need is an image charge  $q''$  outside the sphere at  $z = d$  to simulate the effects of the dielectric material. We should be careful to note that the region has permittivity  $\epsilon$  which should be used in our solution:

$$\Phi_{\text{in}} = \frac{1}{4\pi\epsilon} \left( \frac{q''}{|\mathbf{r} - d\hat{\mathbf{z}}|} \right)$$

The observation point  $r$  is always less than  $d''$ , so there is only one expansion:

$$\Phi_{\text{in}} = \frac{1}{4\pi\epsilon} \left( \sum_{l=0}^{\infty} q'' \frac{r^l}{d^{l+1}} P_l(\cos\theta) \right)$$

Apply the boundary condition at  $r = a$  that  $(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n}_{12} = \sigma$ . There is no free charge density on the sphere's surface, so this boundary condition becomes:

$$(\epsilon_2 \mathbf{E}_2 - \epsilon_1 \mathbf{E}_1) \cdot \mathbf{n}_{12} = 0$$

$$(\epsilon_0 \mathbf{E}_{\text{out}} - \epsilon \mathbf{E}_{\text{in}}) \cdot \hat{\mathbf{r}} = 0$$

$$\epsilon_0 \frac{\partial \Phi_{\text{out}}}{\partial r} = \epsilon \frac{\partial \Phi_{\text{in}}}{\partial r}$$

$$\boxed{q'l + q'(-l-1)\frac{d}{a} = q''l}$$

Apply the boundary condition at  $r = a$  that  $(\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}_{12} = 0$ .

$$E_{\text{in},\theta} = E_{\text{out},\theta}$$

$$\frac{\partial \Phi_{\text{in}}}{\partial \theta} = \frac{\partial \Phi_{\text{out}}}{\partial \theta}$$

$$\boxed{q'' = \frac{\epsilon}{\epsilon_0} q + \frac{\epsilon}{\epsilon_0} q' \frac{d}{a}}$$

Solve the equations in boxes to find:

$$\boxed{q' = \frac{\frac{\epsilon_0}{\epsilon} - 1}{d \left( \frac{\epsilon_0}{\epsilon} (l+1) + l \right)} q l a}$$

$$\boxed{q'' = \frac{2l+1}{\frac{\epsilon_0}{\epsilon} (l+1) + l} q}$$

The final solution is:

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon d} \left( \sum_{l=0}^{\infty} \frac{2l+1}{\frac{\epsilon_0}{\epsilon}(l+1)+l} \left(\frac{r}{d}\right)^l P_l(\cos\theta) \right)$$

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0 d} \sum_{l=0}^{\infty} P_l(\cos\theta) \frac{d^{l+1}}{r^{l+1}} \left( 1 + \frac{\left(\frac{\epsilon_0}{\epsilon}-1\right)l}{\left(\frac{\epsilon_0}{\epsilon}(l+1)+l\right)} \left(\frac{a}{d}\right)^{2l+1} \right) \quad \text{if } r > d$$

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0 d} \sum_{l=0}^{\infty} P_l(\cos\theta) \left( \left(\frac{r}{d}\right)^l + \frac{\left(\frac{\epsilon_0}{\epsilon}-1\right)l}{\left(\frac{\epsilon_0}{\epsilon}(l+1)+l\right)} \left(\frac{a}{d}\right)^l \left(\frac{a}{r}\right)^{l+1} \right) \quad \text{if } r < d$$

(b) Near the center of the sphere, we have  $r \ll d$  or  $r/d \ll 1$ . Because the internal field is in terms of powers of  $r/d$ , the higher order terms become negligible.

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon_0 d} \left[ 1 + \frac{3}{1+2\epsilon_0/\epsilon} \frac{r}{d} \cos\theta \right]$$

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon_0 d} \left[ 1 + \frac{3}{2+\epsilon/\epsilon_0} \frac{z}{d} \right]$$

$$\mathbf{E} = -\nabla\Phi$$

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0 d^2} \left[ \frac{3}{2+\epsilon/\epsilon_0} \right] \hat{\mathbf{z}}$$

The electric field near the center of the sphere points away from the point charge, in the negative  $z$  direction, just as if the sphere was not there, but its magnitude is weakened by the amount in square brackets. For instance, if the dielectric constant of the sphere is 4, then the field near its center is half as strong as if the sphere was not there.

(c) In the limit  $\epsilon/\epsilon_0 \rightarrow \infty$  we can find the fields. We must be careful to treat the  $l=0$  terms separately.

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon_0 d} \left( 1 + \sum_{l=1}^{\infty} \frac{2l+1}{(l+1)+l\epsilon/\epsilon_0} \left(\frac{r}{d}\right)^l P_l(\cos\theta) \right)$$

$$\Phi_{\text{in}} = \frac{q}{4\pi\epsilon_0 d}$$

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0} \frac{1}{r} + \frac{q}{4\pi\epsilon_0 d} \sum_{l=1}^{\infty} P_l(\cos\theta) \frac{d^{l+1}}{r^{l+1}} \left( 1 - \left( \frac{a}{d} \right)^{2l+1} \right) \quad \text{if } r > d$$

$$\Phi_{\text{out}} = \frac{qa}{4\pi\epsilon_0 d r} + \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} P_l(\cos\theta) \frac{d^l}{r^{l+1}} - \frac{q}{4\pi\epsilon_0 d} \sum_{l=0}^{\infty} P_l(\cos\theta) \frac{a^{l+1}}{r^{l+1}} \left( \frac{a^l}{d^l} \right) \quad \text{if } r > d$$

$$\boxed{\Phi_{\text{out}} = \frac{q(a/d)}{4\pi\epsilon_0 r} + \frac{q}{4\pi\epsilon_0 |\mathbf{r} - d\hat{\mathbf{z}}|} - \frac{q(a/d)}{4\pi\epsilon_0 |\mathbf{r} - (a^2/d)\hat{\mathbf{z}}|}} \quad \text{if } r > d$$

$$\Phi_{\text{out}} = \frac{q}{4\pi\epsilon_0 d} + \frac{q}{4\pi\epsilon_0 d} \sum_{l=1}^{\infty} P_l(\cos\theta) \left( \left( \frac{r}{d} \right)^l - \left( \frac{a}{d} \right)^l \left( \frac{a}{r} \right)^{l+1} \right) \quad \text{if } r < d$$

$$\boxed{\Phi_{\text{out}} = \frac{q(a/d)}{4\pi\epsilon_0 r} + \frac{q}{4\pi\epsilon_0 |\mathbf{r} - d\hat{\mathbf{z}}|} - \frac{q(a/d)}{4\pi\epsilon_0 |\mathbf{r} - (a^2/d)\hat{\mathbf{z}}|}} \quad \text{if } r > d$$

This is the same as the potential of point charge near a conducting sphere. The first term is the ungrounded potential before the charge is brought near. The second is due to the point charge and the third is due to the image charge.