Jackson 4.8 Homework Problem Solution

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PROBLEM:

A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field E_0 with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

(a) Determine the potential and electric field in the three regions, neglecting end effects.

(b) Sketch the lines of force for a typical case $b \approx 2a$.

(c) Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

SOLUTION:

(a) Because the cylinder is long and uniform along its axis, and the original field is uniform, the problem reduces down to a two-dimensional polar coordinates problem. Let us place the field pointing in the positive x direction.
There is no free charge anywhere, and there is no bound charge anywhere except on the surface, so we can divide the problem into three regions, and use the solution to the Laplace equation in each region.

Then general solution to the Laplace equation in polar coordinates when the full angular sweep is involved was found to be:

$$\Phi(\rho,\phi)=a_0+b_0\ln\rho+\sum_{m=1}^{\infty}\left(a_m\rho^m+b_m\rho^{-m}\right)\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)$$

The interior region includes the origin, so it must have the form:

$$\boxed{\Phi_{\rho<a}=\sum_{m=0}^{\infty}\rho^m\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)}$$

The external region must become a uniform field very far away:

$$-E_0\rho\cos\phi=a_0+b_0\ln\rho+\sum_{m=1}^{\infty}\left(a_m\rho^m+b_m\rho^{-m}\right)\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)$$

Due to orthogonality, $a_0 = 0$, $b_0 = 0$, only the $m = 1$ term is nonzero, and $A_1 = B_1$

$$-E_0=a_1 2B_1$$

So that the solution becomes:

$$\boxed{\Phi_{\rho>b}=\left(-E_0\rho+b_1\rho^{-1}\right)\cos\phi}$$

The middle region simply connects the other two regions:

$$\boxed{\Phi_{a<\rho<b}=c_0+d_0\ln\rho+\sum_{m=1}^{\infty}\left(c_m\rho^m+d_m\rho^{-m}\right)\left(C_m e^{im\phi}+D_m e^{-im\phi}\right)}$$

Now apply boundary conditions:

$$(\epsilon_2\mathbf{E}_2-\epsilon_1\mathbf{E}_1)\cdot\mathbf{n}=\sigma$$

There is no free charge and the normal direction is in the radial direction:

$$\epsilon_2\mathbf{E}_2\cdot\hat{\boldsymbol{\rho}}=\epsilon_1\mathbf{E}_1\cdot\hat{\boldsymbol{\rho}}$$

$$\epsilon_2\frac{\partial\Phi_2}{\partial\rho}=\epsilon_1\frac{\partial\Phi_1}{\partial\rho}$$

Apply this at the outer surface first ($\rho=b$):

$$\epsilon_0\frac{\partial\Phi_{\rho>b}}{\partial\rho}=\epsilon\frac{\partial\Phi_{a<\rho<b}}{\partial\rho}\quad\text{at }\rho=b$$

$$\epsilon_0\frac{\partial}{\partial\rho}\left(\left(-E_0\rho+b_1\rho^{-1}\right)\cos\phi\right)=\epsilon\frac{\partial}{\partial\rho}\left(c_0+d_0\ln\rho+\sum_{m=1}^{\infty}\left(c_m\rho^m+d_m\rho^{-m}\right)\left(C_m e^{im\phi}+D_m e^{-im\phi}\right)\right)\quad\text{at }\rho=b$$

$$\epsilon_0\left(\left(-E_0-b_1b^{-2}\right)\cos\phi\right)=\epsilon\left(d_0\frac{1}{b}+\sum_{m=1}^{\infty}\left(c_m m b^{m-1}-m d_m b^{-m-1}\right)\left(C_m e^{im\phi}+D_m e^{-im\phi}\right)\right)$$

Due to orthogonality $d_0=0$, $D_m=C_m$ and only the $m=1$ term is nonzero. We can also throw out c_0 as it is just an overall constant that does not effect the final field.

$$\epsilon_0\left(\left(-E_0-b_1b^{-2}\right)\cos\phi\right)=\epsilon\left(\left(c_1-d_1b^{-2}\right)C_1 2\cos\phi\right)$$

The factor c_1 can be combined with C_1

$$\boxed{C_1=\frac{-\epsilon_0\left(E_0+b_1b^{-2}\right)}{2\epsilon\left(1-d_1b^{-2}\right)}}$$

The solution in the middle region now becomes:

$$\boxed{\Phi_{a<\rho<b}=-\left(\rho+d_1\rho^{-1}\right)\frac{\epsilon_0\left(E_0+b_1b^{-2}\right)}{\epsilon\left(1-d_1b^{-2}\right)}\cos\phi}$$

Apply the other boundary condition at the outer surface ($\rho = b$)

$$E_{T,2}=E_{T,1} \quad \text{at } \rho = b$$

$$\frac{\partial \Phi_{\rho>b}}{\partial \phi}=\frac{\partial \Phi_{a<\rho<b}}{\partial \phi} \quad \text{at } \rho = b$$

$$\frac{\partial}{\partial \phi}\left(\left(-E_0\rho+b_1\rho^{-1}\right)\cos\phi\right)=\frac{\partial}{\partial \phi}\left(c_0-\left(\rho+d_1\rho^{-1}\right)\frac{\epsilon_0\left(E_0+b_1b^{-2}\right)}{\epsilon\left(1-d_1b^{-2}\right)}\cos\phi\right) \quad \text{at } \rho = b$$

$$(E_0-b_1b^{-2})\epsilon\left(1-d_1b^{-2}\right)=\left(1+d_1b^{-2}\right)\epsilon_0\left(E_0+b_1b^{-2}\right)$$

$$d_1=b^2\frac{b_1(\epsilon+\epsilon_0)-E_0b^2(\epsilon-\epsilon_0)}{b_1(\epsilon-\epsilon_0)-E_0b^2(\epsilon+\epsilon_0)}$$

The solution in the middle region now becomes:

$$\boxed{\Phi_{a<\rho<b}=\left((b_1(\epsilon-\epsilon_0)-E_0b^2(\epsilon+\epsilon_0))\rho+b^2(b_1(\epsilon+\epsilon_0)-E_0b^2(\epsilon-\epsilon_0))\rho^{-1}\right)\frac{1}{2\epsilon b^2}\cos\phi}$$

Now apply boundary conditions at the inner surface ($\rho = a$):

$$\epsilon\frac{\partial \Phi_{a<\rho<b}}{\partial \rho}=\epsilon_0\frac{\partial \Phi_{\rho<a}}{\partial \rho} \quad \text{at } \rho = a$$

$$\left((b_1(\epsilon-\epsilon_0)-E_0b^2(\epsilon+\epsilon_0))-b^2(b_1(\epsilon+\epsilon_0)-E_0b^2(\epsilon-\epsilon_0))a^{-2}\right)\frac{1}{2b^2}=\epsilon_0A_1$$

As well as:

$$\frac{\partial \Phi_{a<\rho<b}}{\partial \phi}=\frac{\partial \Phi_{\rho<a}}{\partial \phi} \quad \text{at } \rho = a$$

$$\left((b_1(\epsilon-\epsilon_0)-E_0b^2(\epsilon+\epsilon_0))a+b^2(b_1(\epsilon+\epsilon_0)-E_0b^2(\epsilon-\epsilon_0))a^{-1}\right)\frac{1}{2\epsilon ab^2}=A_1$$

Solve this system of equations for the remaining constants:

$$\boxed{b_1=E_0b^2\frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}}$$

$$\boxed{A_1=\left(\frac{-4b^2E_0\epsilon\epsilon_0}{(b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2)}\right)}$$

So that the final solutions are:

$$\boxed{\Phi_{\rho<a}=\left(\frac{-4b^2\epsilon\epsilon_0}{(b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2)}\right)E_0\rho\cos\phi}$$

$$\boxed{\Phi_{a<\rho<b}=\frac{-2ab^2\epsilon_0}{(b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2)}\left((\epsilon+\epsilon_0)\frac{\rho}{a}+(\epsilon-\epsilon_0)\frac{a}{\rho}\right)E_0\cos\phi}$$

$$\boxed{\Phi_{\rho>b}=\left(-\rho+\frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{\rho}\right)E_0\cos\phi}$$

Let us calculate the electric fields:

$$\mathbf{E}=-\hat{\boldsymbol{\rho}}\frac{\partial\Phi}{\partial\rho}-\hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}$$

$$\mathbf{E}_{\rho<a}=\left(\frac{4b^2\epsilon\epsilon_0}{(b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2)}\right)E_0\left[\hat{\boldsymbol{\rho}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi\right]$$

$$\boxed{\mathbf{E}_{\rho<a}=\left(\frac{4b^2\epsilon\epsilon_0}{(b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2)}\right)E_0\hat{\mathbf{i}}}$$

$$\boxed{\mathbf{E}_{a<\rho<b}=\frac{2b^2\epsilon_0}{(b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2)}E_0\left[(\epsilon+\epsilon_0)\hat{\mathbf{i}}-(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}(\hat{\mathbf{i}}+2\hat{\boldsymbol{\phi}}\sin\phi)\right]}$$

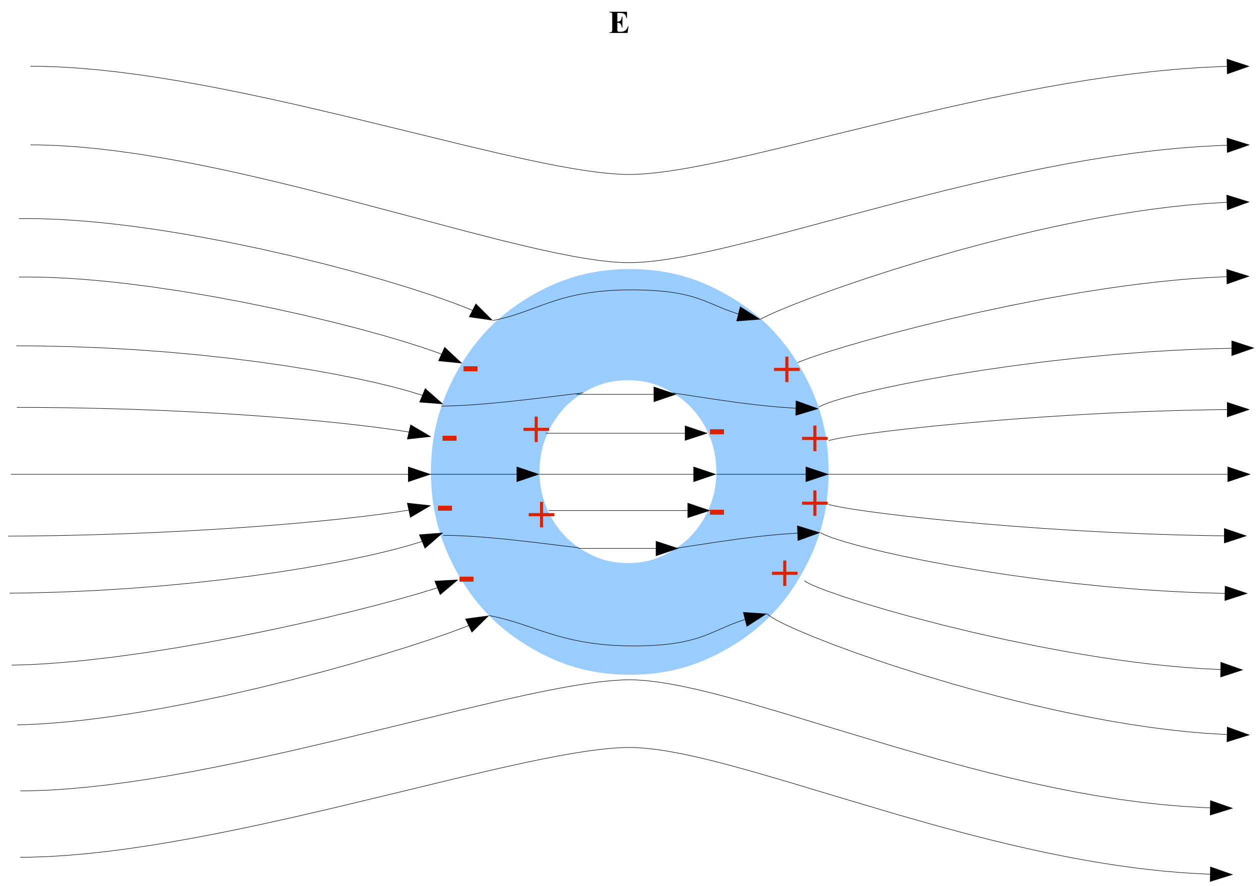
$$\boxed{\mathbf{E}=E_0\hat{\mathbf{i}}+\frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{\rho^2}E_0(\hat{\mathbf{i}}+2\hat{\boldsymbol{\phi}}\sin\phi)}$$

(b) Let us sketch the lines of force for the typical case $b\approx 2a$.

$$\boxed{\mathbf{E}_{\rho<a}=\left(\frac{16\epsilon\epsilon_0}{(4(\epsilon+\epsilon_0)^2-(\epsilon-\epsilon_0)^2)}\right)E_0\hat{\mathbf{i}}}$$

$$\boxed{\mathbf{E}_{a<\rho<b}=\frac{8\epsilon_0}{(4(\epsilon+\epsilon_0)^2-(\epsilon-\epsilon_0)^2)}E_0\left[(\epsilon+\epsilon_0)\hat{\mathbf{i}}-(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}(\hat{\mathbf{i}}+2\hat{\boldsymbol{\phi}}\sin\phi)\right]}$$

$$\boxed{\mathbf{E}_{\rho>b}=E_0\hat{\mathbf{i}}+\frac{12(\epsilon^2-\epsilon_0^2)}{4(\epsilon+\epsilon_0)^2-(\epsilon-\epsilon_0)^2}\frac{a^2}{\rho^2}E_0(\hat{\mathbf{i}}+2\hat{\boldsymbol{\phi}}\sin\phi)}$$



Note that the field is uniform in the interior. The left outside edge develops a negative surface bound charge density, which destroys field lines. The right outside edge develops a positive surface bound charge density, which creates field lines. As a result there are less field lines, and thus a weaker field inside the material. The negative bound charges in the left outside would attract a test charge, so the field lines are bent towards the object.

(c) For a solid dielectric cylinder in a uniform field, we simply let a approach zero.

$$\boxed{\mathbf{E}_{\rho<b}=\frac{2\epsilon_0}{(\epsilon+\epsilon_0)}E_0\hat{\mathbf{i}}}$$

$$\boxed{\mathbf{E}_{\rho>b}=E_0\hat{\mathbf{i}}+\frac{(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)}\frac{b^2}{\rho^2}E_0(\hat{\mathbf{i}}+2\hat{\boldsymbol{\phi}}\sin\phi)}$$

When the cylinder becomes solid, the field inside becomes uniform.

For a cylindrical cavity in a uniform dielectric, we let b approach infinity.

$$\boxed{\mathbf{E}_{\rho<a} = \frac{4\,\epsilon\,\epsilon_0}{(\epsilon+\epsilon_0)^2}\,E_0\,\hat{\mathbf{i}}}$$

$$\boxed{\mathbf{E}_{a<\rho} = \frac{2\,\epsilon_0}{(\epsilon+\epsilon_0)^2}\,E_0\left[(\epsilon+\epsilon_0)\,\hat{\mathbf{i}} - (\epsilon-\epsilon_0)\,\frac{a^2}{\rho^2}\,(\hat{\mathbf{i}} + 2\,\hat{\boldsymbol{\phi}}\sin\phi)\right]}$$