Jackson 4.5 Homework Problem Solution

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PROBLEM:

A localized charge density $\rho(x, y, z)$ is placed in an external electrostatic field described by a potential $\Phi^{(0)}(x, y, z)$. The external potential varies slowly in space over a region where the charge density is different from zero.

(a) From first principles calculate the total *force* acting on the charge distribution as an expansion in multipole moments times derivatives of the electric field, up to and including the quadrupole moments. Show that the force is

$$\mathbf{F} = q\,\mathbf{E}^{(0)} + (\nabla[\mathbf{p}\cdot\mathbf{E}^{(0)}])_0 + \left(\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^{(0)}}{\partial x_k}(\mathbf{x})\right]\right)_0 + \ldots$$

Compare this to the expansion (4.24) of the *energy W*. Note that (4.24) is a number – it is not a function of **x** that can be differentiated! What is its connection to **F**?

(b) Repeat the calculation of part a for the total *torque*. For simplicity, evaluate only one Cartesian component of the torque, say N_1. Show that this component is:

$$N_1 = [\mathbf{p}\times\mathbf{E}^{(0)}(0)]_1 + \frac{1}{3}\left[\frac{\partial}{\partial x_3}\left(\sum_j Q_{2j}E_j^{(0)}\right) - \frac{\partial}{\partial x_2}\left(\sum_j Q_{3j}E_j^{(0)}\right)\right]_0 + \ldots$$

SOLUTION:

(a) The force in general is:

$$\mathbf{F} = \int \rho(\mathbf{x})\mathbf{E}^{(0)}(\mathbf{x})\,d^3\mathbf{x}$$

Let us keep track of the components.

$$\mathbf{F} = \sum_i \hat{\mathbf{x}}_i \int \rho(\mathbf{x}) E_i^{(0)}(\mathbf{x})\,d^3\mathbf{x}$$

Expand the external electric field in a Taylor series and only keep the focus on the first few terms because it varies slowly over space:

$$E_i^{(0)}(\mathbf{x}) = \left[E_i^{(0)}(\mathbf{x}') + \sum_j x_j \frac{\partial}{\partial x_j'} E_i^{(0)}(\mathbf{x}') + \frac{1}{2}\sum_{j,k} x_j x_k \frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_k'} E_i^{(0)}(\mathbf{x}') + \ldots\right]_{\mathbf{x}'=0}$$

Note we have labeled the gradients as being done with respect to primed variables to tell them apart from the integration variables. Substitute the expansion into the force equation:

$$\mathbf{F}=\left[\mathbf{E}^{(0)}(\mathbf{x}')q+\sum_i\sum_j\hat{\mathbf{x}}_i\int\rho(\mathbf{x})x_j d^3\mathbf{x}\frac{\partial}{\partial x_j'}E_i^{(0)}+\frac{1}{2}\sum_i\sum_{j,k}\hat{\mathbf{x}}_i\int\rho(\mathbf{x})x_j x_k d^3\mathbf{x}\frac{\partial}{\partial x_j'}\frac{\partial}{\partial x_k'}E_i+...\right]_{\mathbf{x}'=0}$$

Note that $\nabla\times\mathbf{E}=0$ means that $\frac{\partial E_i}{\partial x_j}=\frac{\partial E_j}{\partial x_i}$. We use this on the dipole terms and the quadrupole terms:

$$\mathbf{F}=\left[\mathbf{E}^{(0)}(\mathbf{x}')q+\sum_i\sum_j\hat{\mathbf{x}}_i\int\rho(\mathbf{x})x_j d^3\mathbf{x}\frac{\partial}{\partial x_i'}E_j^{(0)}+\frac{1}{2}\sum_i\sum_{j,k}\hat{\mathbf{x}}_i\int\rho(\mathbf{x})x_j x_k d^3\mathbf{x}\frac{\partial}{\partial x_i'}\frac{\partial}{\partial x_k'}E_j^{(0)}+...\right]_{\mathbf{x}'=0}$$

The electric fields do not depend on the unprimed variables and come out of the integrals, which was the point of the Taylor series expansion. After a little manipulation, we recognize the integrals that are left as the dipole moment and quadrupole moments:

$$\mathbf{F}=\left[\mathbf{E}^{(0)}(\mathbf{x}')q+\sum_i\hat{\mathbf{x}}_i\frac{\partial}{\partial x_i'}\mathbf{p}\cdot\mathbf{E}^{(0)}+\frac{1}{6}\sum_i\hat{\mathbf{x}}_i\frac{\partial}{\partial x_i'}\left(\sum_{j,k}Q_{jk}\frac{\partial}{\partial x_k'}E_j^{(0)}+\sum_j\frac{\partial}{\partial x_j'}E_j^{(0)}\int\rho(\mathbf{x})r^2 d^3\mathbf{x}\right)+...\right]_{\mathbf{x}'=0}$$

Next note that the external field is created by charges outside our volume of interest, so:

$$\nabla\cdot\mathbf{E}^{(0)}=0\quad\text{or}\quad\sum_i\frac{\partial E_i^{(0)}}{\partial x_i}=0$$

This is the exact factor found in the last term shown, dropping that entire term out. Replacing the index notation with vector notation, we finally have:

$$\boxed{\mathbf{F}=q\,\mathbf{E}^{(0)}(0)+(\nabla[\mathbf{p}\cdot\mathbf{E}^{(0)}])_0+\left(\nabla\left[\frac{1}{6}\sum_{j,k}Q_{jk}\frac{\partial E_j^{(0)}}{\partial x_k}(\mathbf{x})\right]\right)_0+...}$$

Here we have switched primed variables to unprimed to match Jackson, and because the originally unprimed variables (the integration variables) are neatly tucked away now in the multipole moments.

If we write the first term in terms of the potential, we can factor out the gradient operator:

$$\mathbf{F}=-\nabla\left[q\,\Phi^{(0)}-\mathbf{p}\cdot\mathbf{E}^{(0)}-\frac{1}{6}\sum_{j,k}Q_{jk}\frac{\partial E_j^{(0)}}{\partial x_k}(\mathbf{x})+...\right]_0$$

Let us compare this to the expansion (Jackson 4.24) of the *energy W*:

$$W=q\,\Phi(0)-\mathbf{p}\cdot\mathbf{E}(0)-\frac{1}{6}\sum_i\sum_j Q_{ij}\frac{\partial E_j}{\partial x_i}(0)+...$$

We see that using both equations, we recover the familiar expression:

$$\boxed{W=-\int\mathbf{F}\cdot d\,\mathbf{x}_{(0)}}$$

(b) The electrostatic torque on a charge distribution ρ as a result of the external field $\mathbf{E}^{(0)}$ is:

$$\mathbf{N}=\int \mathbf{x}\times(\rho(\mathbf{x})\mathbf{E}^{(0)}(\mathbf{x}))d^3\mathbf{x}$$

Let us look at one component, say component 1:

$$N_1=\int \rho(\mathbf{x})(x_2 E_3^{(0)}-x_3 E_2^{(0)})d^3\mathbf{x}$$

Expand the electric field in a Taylor series:

$$E_2^{(0)}(\mathbf{x})=\left[E_2^{(0)}(\mathbf{x}')+\sum_j x_j \frac{\partial}{\partial x_j'}E_2^{(0)}(\mathbf{x}')+...\right]_{\mathbf{x}'=0}$$

and

$$E_3^{(0)}(\mathbf{x})=\left[E_3^{(0)}(\mathbf{x}')+\sum_j x_j \frac{\partial}{\partial x_j'}E_3^{(0)}(\mathbf{x}')+...\right]_{\mathbf{x}'=0}$$

and insert these into the torque equation to find:

$$N_1=[E_3^{(0)}(\mathbf{x}')\int \rho(\mathbf{x})x_2 d^3\mathbf{x}-E_2^{(0)}(\mathbf{x}')\int \rho(\mathbf{x})x_3 d^3\mathbf{x}$$
$$+\sum_j \frac{\partial}{\partial x_j'}E_3^{(0)}\int \rho(\mathbf{x})x_2 x_j d^3\mathbf{x}-\sum_j \frac{\partial}{\partial x_j'}E_2^{(0)}\int \rho(\mathbf{x})x_3 x_j d^3\mathbf{x}\,]_{\mathbf{x}'=0}+...$$

We have moved the electric field components out of the integrals because they do not depend on the unprimed integration variables. This was the point of the Taylor expansion. Just as was done in the previous section, the non-diverging nature of the external electric field means that there is a piece equal to zero that we can add to get the last terms to look like the quadrupole moments. We end up with:

$$N_1=[E_3^{(0)}(\mathbf{x}')p_2-E_2^{(0)}(\mathbf{x}')p_3+\frac{1}{3}\sum_j \frac{\partial}{\partial x_j'}E_3^{(0)}Q_{2j}-\frac{1}{3}\sum_j \frac{\partial}{\partial x_j'}E_2^{(0)}Q_{3j}]_{\mathbf{x}'=0}+...$$

Again use the relation: $\dfrac{\partial E_i}{\partial x_j}=\dfrac{\partial E_j}{\partial x_i}$

$$N_1=[E_3^{(0)}(\mathbf{x}')p_2-E_2^{(0)}(\mathbf{x}')p_3+\frac{1}{3}\frac{\partial}{\partial x_3'}\sum_j E_j^{(0)}Q_{2j}-\frac{1}{3}\frac{\partial}{\partial x_2'}\sum_j E_j^{(0)}Q_{3j}]_{\mathbf{x}'=0}+...$$

$$\boxed{N_1=[\mathbf{p}\times\mathbf{E}^{(0)}(0)]_1+\frac{1}{3}\left[\frac{\partial}{\partial x_3}\left(\sum_j Q_{2j}E_j^{(0)}\right)-\frac{\partial}{\partial x_2}\left(\sum_j Q_{3j}E_j^{(0)}\right)\right]_0+...}$$