Jackson 4.4 Homework Problem Solution

Dr. Christopher S. Baird
University of Massachusetts Lowell

PROBLEM:

(a) Prove the following theorem: For an arbitrary charge distribution $\rho(\mathbf{x})$ the values of the $(2l + 1)$ moments of the first non-vanishing multipole are independent of the origin of the coordinate axes, but the values of all higher multipole moments do in general depend on the choice of origin. (The different moments q_{lm} for fixed l depend, of course, on the orientation of the axes.)

(b) A charge distribution has multipole moments q, $\mathbf{p}$, Q_{ij}, … with respect to one set of coordinate axes, and moments q', $\mathbf{p}'$, Q'_{ij}, … with respect to another set whose axes are parallel to the first, but whose origin is located at the point $\mathbf{R} = (X, Y, Z)$ relative to the first. Determine explicitly the connections between the monopole, dipole, and quadrupole moments in the two coordinate frames.

(c) If $q \neq 0$, can $\mathbf{R}$ be found so that $\mathbf{p}' = 0$? If $q \neq 0$, $\mathbf{p} \neq 0$, or at least $\mathbf{p} \neq 0$, can R be found so that $Q'_{ij} = 0$?

SOLUTION:

(a) Some point is located at $\mathbf{x} = (x, y, z)$ relative to an initial origin. If we choose a new origin that is located at $\mathbf{R} = (X, Y, Z)$, then the point in this new coordinate system will be at $\mathbf{x}' = \mathbf{x} - \mathbf{R}$, or $(x', y', z') = (x - X, y - Y, z - Z)$.

In general, the lth set of moments is given by (we ignore the traceless nature):

$$Q_{ij...l} = \int\int\int \rho(x, y, z)\, x_i x_j \ldots x_l \, dx\, dy\, dz$$

where $i = 1, 2, 3$ and $j = 1, 2, 3$, and etc. and $x_1 = x$, $x_2 = y$, $x_3 = z$.
The moments in this initial coordinate system in terms of the new coordinate system are:

$$Q_{ijk...l} = \int\int\int \rho(x'+X, y'+Y, z'+Z)(x'_i + X_i)(x'_j + X_j)\ldots(x'_l + X_l)\, dx'\, dy'\, dz'$$

Expand out all the terms, recognizing that the components of $\mathbf{R}$ are constant and come out of the integral:

$$
\begin{aligned}
Q_{ijk...l} = & \int\int\int \rho(x'+X, y'+Y, z'+Z) x'_i x'_j ... x'_l dx' dy' dz' \\
& + X_i \int\int\int \rho(x'+X, y'+Y, z'+Z) x'_j ... x'_l dx' dy' dz' \\
& + X_j \int\int\int \rho(x'+X, y'+Y, z'+Z) x'_i ... x'_l dx' dy' dz' \\
& + ... \\
& + X_l \int\int\int \rho(x'+X, y'+Y, z'+Z) x'_j ... x'_{l-1} dx' dy' dz' \\
& + X_i X_j \int\int\int \rho(x'+X, y'+Y, z'+Z) x'_k ... x'_l dx' dy' dz' \\
& + ... \\
& + X_i X_j ... X_l \int\int\int \rho(x'+X, y'+Y, z'+Z) dx' dy' dz'
\end{aligned}
$$

Now recognize that the first integral is just the multipole in the new coordinate system, and the other terms are lower order multipoles in the new coordinate system.

$$
\begin{aligned}
Q_{ijk...l} = & Q'_{ijk...l} + X_i Q'_{jk...l} + X_j Q'_{ik...l} + ... + X_l Q'_{ijk...l-1} \\
& + X_i X_j Q'_{k...l} + ... + X_{l-1} X_l Q'_{ijk...l-2} \\
& + ... \\
& + X_i X_j ... X_l Q'
\end{aligned}
$$

If the multiple moments are to have the same value in both coordinate systems, all remaining terms must be zero. In other words:

$$
Q_{ijk...l} = Q'_{ijk...l} \quad \text{only if} \quad
\begin{aligned}
0 = & X_i Q'_{jk...l} + X_j Q'_{ik...l} + ... + X_l Q'_{ijk...l-1} \\
& + X_i X_j Q'_{k...l} + ... + X_{l-1} X_l Q'_{ijk...l-2} \\
& + ... \\
& + X_i X_j ... X_l Q'
\end{aligned}
$$

This can only hold true for all arbitrary X_i, X_j, etc if all terms are independently zero:

$$
\boxed{Q_{ijk...l} = Q'_{ijk...l} \quad \text{only if} \quad Q'_{ijk...l-1} = 0, \qquad Q'_{ijk...l-2} = 0 \quad ...}
$$

In words, this says that the value of the multipole moment of order l is independent of origin choice *if and only if* all lower order multipole moments are zero.

(b) This is just a specific application of the general transformation above.
For a monopole, $l = 0$, the general form:

$$
Q_{ij...l} = \int\int\int \rho(x, y, z) x_i x_j ... x_l \, dx \, dy \, dz
$$

becomes:

$$
q = \int\int\int \rho(x, y, z) \, dx \, dy \, dz
$$

In the new coordinate system, this is:

$$
q = \int\int\int \rho(x'+X, y'+Y, z'+Z) dx' dy' dz'
$$

$$\boxed{q = q'}$$

For a dipole, $l = 1$, the general form reduces down to:

$p_i = \int \int \int \rho(x, y, z) x_i \, dx \, dy \, dz$ for $i = 1, 2, 3$ such that $p_1 = p_x$, $p_2 = p_y$, $p_3 = p_z$

In the new coordinate system, this is:

$$p_i = \int \int \int \rho(x'+X, y'+Y, z'+Z)(x'_i + X_i) dx' dy' dz'$$

$$p_i = \int \int \int \rho(x'+X, y'+Y, z'+Z) x'_i dx' dy' dz' + X_i \int \int \int \rho(x'+X, y'+Y, z'+Z) dx' dy' dz'$$

$$p_i = p_i' + X_i q'$$

$$\boxed{\mathbf{p} = \mathbf{p}' + \mathbf{R} q'}$$

For a quadrupole, $l = 2$, the general form reduces down to:

$$Q_{ij} = \int \int \int \rho(x, y, z) x_i x_j \, dx \, dy \, dz$$

To follow the convention of the book, we use the traceless versions:

$$Q_{ij} = \int \int \int \rho(x, y, z)(3 x_i x_j - r^2 \delta_{ij}) dx \, dy \, dz$$

In the new coordinate system, this is:

$$Q_{ij} = \int \int \int \rho(x'+X, y'+Y, z'+Z)(3(x_i + X_i)(x_j + X_j) - ((x'+X)^2 + (y'+Y)^2 + (z'+Z)^2)\delta_{ij}) dx' dy' dz'$$

$$Q_{ij} = Q'_{ij} + 3 X_j p'_i + 3 X_i p'_j + 3 X_i X_j q' - R^2 \delta_{ij} q' - 2 \delta_{ij} \mathbf{R} \cdot \mathbf{p}'$$

$$\boxed{Q_{ij} = Q'_{ij} + 3 X_j p'_i + 3 X_i p'_j - 2 \delta_{ij} \mathbf{R} \cdot \mathbf{p}' + (3 X_i X_j - R^2 \delta_{ij}) q'}$$

(c) If $q \neq 0$, then $\mathbf{p}' = 0$ if:

$$\boxed{\mathbf{R} = \frac{\mathbf{p}}{q}}$$

For instance, consider the simple charge distribution consisting of a point charge $+A$ at $z = a$, a point charge -1 at $z = -A$, and a point charge $+2A$ at the origin. In this coordinate system, the total charge is $q = +2A$ and the dipole moment is $p = 2Aa$. If we now place the origin at $z = a$, this system will have no dipole moment.

If $q = 0$, $\mathbf{p} \neq 0$, then $Q'_{ij} = 0$ if:

$$Q_{ij} = 3 X_j p_i + 3 X_i p_j - 2 \delta_{ij} \mathbf{R} \cdot \mathbf{p}$$

Writing this out and solving the system of equations leads to:

$$X_1 = \frac{(-Q_{23} p_1 + Q_{13} p_2 + Q_{12} p_3)}{6 p_2 p_3}$$

$$X_2 = \frac{(Q_{23} p_1 - Q_{13} p_2 + Q_{12} p_3)}{6 p_1 p_3}$$

$$X_3 = \frac{Q_{23} p_1 + Q_{13} p_2 - Q_{12} p_3}{6 p_1 p_2}$$